Math 251:12–17 # Workshop Problems #1 **October 13, 2010**

1. Each of four real numbers is known to lie between 0 and 50. The product of these four numbers is approximated by first rounding each number to the nearest tenth (that is, to the nearest number with one digit after the decimal point) and then taking the product of these rounded numbers. Use differentials (linear approximation) to estimate the maximum possible error in the approximate product.

2. The 2×2 determinant can be thought of as a function which takes four variables as input, and returns a real number as output:

$$\det(a, b, c, d) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

a) What is the gradient of this function, $\nabla \det$? (The gradient of any function is a vector. First question: how many components will $\nabla \det$ have?)

b) If $a = 2$, $b = -3$, $c = 4$, and $d = 5$, then

$$\det(a, b, c, d) = \det(2, -3, 4, 5) = \det \begin{pmatrix} 2 & -3 \\ 4 & 5 \end{pmatrix} = 22.$$

Suppose we want to change each of a, b, c, and d by a little bit, where "little bit" here means that $(\triangle a)^2 + (\triangle b)^2 + (\triangle c)^2 + (\triangle d)^2 \leq .01$. If we want to make changes so the new determinant is as *large* as possible, what changes would you recommend?

The Second Derivative Test for functions of two variables is useful when considering the next two problems. You can read about it in the text, but the following statement of the result should be sufficient.

The Second Derivative Test for functions of two variables

Suppose that $f(x, y)$ is a differentiable function of x and y, and that (x_0, y_0) is a critical point of f (that is, $\nabla f(x_0, y_0) = 0$, or both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$). Compute $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$.

- If $D > 0$ and if $f_{xx}(x_0, y_0) > 0$ then $f(x, y)$ has a **local minimum** at (x_0, y_0).
- If $D > 0$ and if $f_{xx}(x_0, y_0) < 0$ then $f(x, y)$ has a **local maximum** at (x_0, y_0).
- If $D < 0$ then $f(x, y)$ has a **saddle point** at (x_0, y_0).

  If $D = 0$ then this result supplies *no information*.

Examples $x^2 + y^2$ and $-x^2 - y^2$ both have c.p.'s at $(0, 0)$ and also $D = 4 > 0$ there. The test can be used to conclude that the first function has a local min at $(0, 0)$ and the second function has a local max at $(0, 0)$. $x^2 - y^2$ also has a c.p. at $(0, 0)$ with $D = -4 < 0$, so $x^2 - y^2$ has a saddle point at $(0, 0)$.

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3. Suppose the line L_1 is $\begin{cases} x = 2t+1 \\ y = -t-1 \\ z = 3t \end{cases}$ and the line L_2 is $\begin{cases} x = 3s+2 \\ y = 5s-2 \\ z = -4 \end{cases}$. Define the function $f(s,t)$ to be the distance between the point on line L_1 with parameter value s and the point on the line L_2 with parameter value t.

a) Find and classify (max/min/saddle) all critical points of $f(s,t)$. (There is exactly one!)

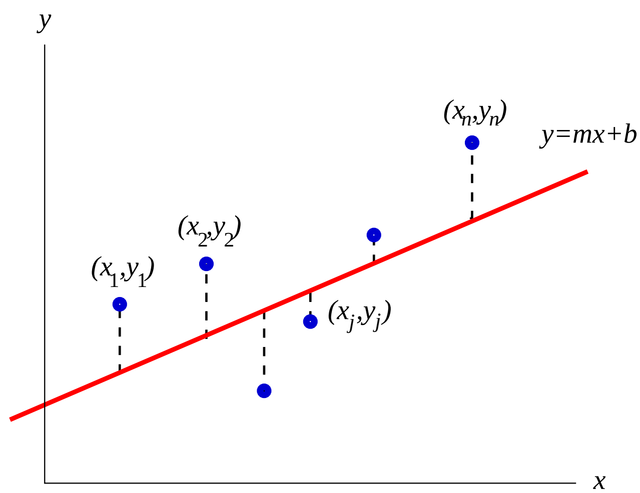
b) The line segment which has endpoints characterized by the values of s and t discovered in a) has an interesting geometric property related to L_1 and L_2. What is this property? Use a drawing to help your explanation.

4. Given n data points $(x_1, y_1), \ldots, (x_n, y_n)$, we may seek a linear function $y = mx + b$ that best fits the data. The **linear least-squares fit** is the linear function $f(x) = mx + b$ that minimizes the sum of the squares (see the Figure) $E(m,b) = \sum_{nj=1}^{n} (y_j - f(x_j))^2$.

Show that E is minimized for m and b satisfying

$$m \sum_{j=1}^{n} x_j + bn = \sum_{j=1}^{n} y_j \text{ and } m \sum_{j=1}^{n} x_j^2 + b \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} x_j y_j$$

Comment This is problem 44 in the textbook's section 14.7. The result is quite important in practical computation. Several assertions must be verified: that E has one critical point which is a local minimum, and that this local minimum is actually an *absolute* minimum.



One problem will be selected for a writeup to be handed in at the next recitation meeting. Please see Professor Greenfield's Math 251 webpage to learn which problem to hand in.

Your workshop writeup will be read either by the lecturer or the recitation instructor. Grading will be on a 10 point scale: 5 points for mathematical content and 5 points for exposition which should be in *complete English sentences*. Neatness counts! Further explanation of what is desired will be linked to the course webpage.