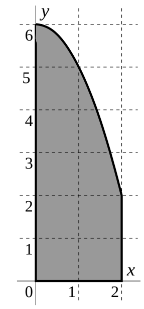
Here are answers to Version A. Other methods may also be correct.

(14) 1. This problem investigates the iterated integral $I = \int_0^2 \int_0^{6-x^2} x\,dy\,dx$.



a) Compute I. **Answer** $\int_0^2 \int_0^{6-x^2} x\,dy\,dx = \int_0^2 xy\big|_{y=0}^{y=6-x^2}\,dx = \int_0^2 x(6-x^2)\,dx = \int_0^2 6x - x^3\,dx = 3x^2 - \frac{1}{4}x^4\Big|_0^2 = 12 - \frac{16}{4} = 8.$

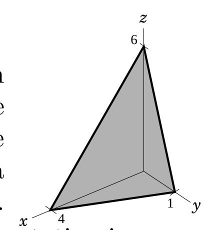
b) Use the axes to the right to sketch the region of integration for I. **Answer** Shown to the right.

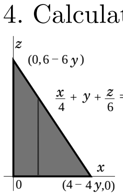
c) Write I as a sum of one or more $dx\,dy$ integrals. You do not need to compute the result!
Answer $\int_0^2 \int_0^2 x\,dx\,dy + \int_2^6 \int_0^{\sqrt{6-y}} x\,dx\,dy$.

(10) 2. Find and classify using the Second Derivative Test all critical points of $f(x,y) = x^2 - y + \ln(3x+y)$.
Answer Since $f_x = 2x + \frac{3}{3x+y}$ and $f_y = -1 + \frac{1}{3x+y}$ we know that $-1 + \frac{1}{3x+y} = 0$ so $3x + y = 1$. Therefore $f_x = 0$ becomes $2x + 3 = 0$ so that $x = -\frac{3}{2}$. Then $y = 1 - 3x = \frac{11}{2}$.
Now for the Second Derivative Test. We compute $f_{xx} = 2 - \frac{9}{(3x+y)^2}$, $f_{xy} = -\frac{3}{(3x+y)^2}$, $f_{yx} = -\frac{3}{(3x+y)^2}$, and $f_{yy} = -\frac{1}{(3x+y)^2}$. Then at the critical point, $D = f_{xx}f_{yy} - (f_{xy})^2 = (-7)(-1) - 3^2 = -2$. This critical point is a saddle point.

(12) 3. Use the Lagrange multiplier method to find the maximum and minimum values of $f(x,y,z) = 3x - y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.
Answer The constraint is $g(x,y,z) = x^2 + y^2 + z^2 = 1$. The Lagrange multiplier equations in vector form are $\nabla f = \lambda \nabla g$ and these are three equations which we must solve together with the constraint equation: $3 = \lambda(2x)$, $-1 = \lambda(2y)$, $2 = \lambda(2z)$, and $x^2 + y^2 + z^2 = 1$. The first three equations show that none of the variables λ, x, y, and z may be 0 in any solution because the left-hand sides are all non-zero. Then we can solve those equations to get $x = \frac{3}{2\lambda}$, $y = -\frac{1}{2\lambda}$, and $z = \frac{2}{2\lambda}$. The constraint equation then becomes $\left(\frac{3}{2\lambda}\right)^2 + \left(-\frac{1}{2\lambda}\right)^2 + \left(\frac{2}{2\lambda}\right)^2 = 1$ so that $\lambda^2 = \frac{7}{2}$ and $\lambda = \pm\sqrt{\frac{7}{2}}$. The values of the objective function are then $f(x,y,z) = 3x - y + 2z = \frac{9}{2\lambda} + \frac{1}{2\lambda} + \frac{4}{2\lambda} = \frac{14}{2\lambda} = \pm\sqrt{14}$. The maximum value is $\sqrt{14}$ and the minimum value is $-\sqrt{14}$.

(16) 4. Calculate the integral of $f(x,y,z) = y$ over the tetrahedron W shown to the right.



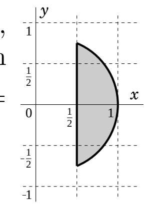


Answer The "tilted" side of the tetrahedron has equation $\frac{x}{4} + y + \frac{z}{6} = 1$. Then the triple integral can be converted into an iterated integral. The y limits are $y = 0$ and $y = 1$. A typical "intermediate" y slice (for $0 < y < 1$ is shown to the left. The x limits are $x = 0$ and $x = 4 - 4y$. Inside the y slice we can look at a typical slice when x is constant. Then z varies from $z = 0$ to $z = 6\left(1 - \frac{x}{4} - y\right)$.
The tilted line is the intersection of the tilted side with the constant y slice. And the computation is:
$\iiint_W y\,dV = \int_0^1 \int_0^{4-4y} \int_0^{6-(6/4)x-6y} y\,dz\,dx\,dy = \int_0^6 \int_0^{4-4y} y\left(6 - \frac{6}{4}x - 6y\right)\,dx\,dy = \int_0^6 \int_0^{4-4y} 6y - \frac{3}{2}xy - 6y^2\,dx\,dy$.
The inner integral: $\int_0^{4-4y} 6y - \frac{3}{2}xy - 6y^2\,dx = 6(4-4y)y - \frac{3}{4}(4-4y)^2 y - 6y^2(4-4y) = 12y - 24y^2 + 12y^3$.
Finally, we compute $\int_0^1 12y - 24y^2 + 12y^3\,dy = 6y^2 - 8y^3 + 3y^4\Big|_0^1 = 1$.
I think the computation is uglier algebraically if y is more "inside" the iterated integral.

OVER

(14) 5. Sketch the D indicated on the axes provided and integrate $f(x,y)$ over D using polar coordinates.

$$f(x,y)=x(x^2+y^2)^{-1};\quad x\geq\frac{1}{2},\quad x^2+y^2\leq 1$$



Answer The sketch is to the right. Since $x=r\cos\theta$, $x=\frac{1}{2}$ becomes $r=\frac{1}{2\cos\theta}$. Also, $x^2+y^2=1$ becomes $r=1$. The limits on θ are $-\frac{\pi}{3}$ and $\frac{\pi}{3}$ (well-known triangles are hidden in the picture). So $\iint_D f(x,y)\,dA=\int_{-\pi/3}^{\pi/3}\int_{1/(2\cos\theta)}^{1}\frac{r\cos\theta}{r^2}\,r\,dr\,d\theta=\int_{-\pi/3}^{\pi/3}\int_{1/(2\cos\theta)}^{1}\cos\theta\,dr\,d\theta=\int_{-\pi/3}^{\pi/3}(\cos\theta)r\Big|_{r=1/(2\cos\theta)}^{r=1}d\theta=\int_{-\pi/3}^{\pi/3}\left(\cos\theta-\frac{1}{2}\right)d\theta=\sin\theta-\frac{1}{2}\theta\Big|_{-\pi/3}^{\pi/3}=\sqrt{3}-\frac{\pi}{3}$

(12) 6. Use cylindrical coordinates to find the mass of a cylinder of radius 4 and height 7 if the mass density at a point is equal to the square of the distance from the cylinder's central axis.

Answer The mass is the triple integral of the density over the cylinder. The cylinder's axis of symmetry will be the z axis, and the cylinder will extend from $z=0$ to $z=7$. The density will be r^2. The mass is $\int_0^7\int_0^{2\pi}\int_0^4 r^2\,(r\,dr\,d\theta)\,dz=\frac{4^4}{4}\cdot(2\pi)\cdot 7$, a fine answer.

(12) 7. A three dimensional region R is those points in <u>the first octant</u> with distance to the origin <u>between 2 and 3</u>. Compute the triple integral of z^2 over this region. **Suggestion** Spherical.

Answer Since $z=\rho\,\cos\phi$, the integrand is $\rho^2(\cos\phi)^2$. Also $dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, and the limits in spherical coordinates are easy. The triple integral to be computed is $\int_0^{\pi/2}\int_0^{\pi/2}\int_2^3\left(\rho^2(\cos\phi)^2\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$. So we compute: $\int_0^{\pi/2}\int_0^{\pi/2}\int_2^3\rho^4(\cos\phi)^2\sin\phi\,d\rho\,d\phi\,d\theta=\left(\frac{3^5-2^5}{5}\right)\cdot\frac{1}{3}\cdot\frac{\pi}{2}$, another fine answer.

(10) 8. This problem is about the transformation $\begin{cases}x=u^2-v^2\\ y=2uv\end{cases}$.

a) Compute the Jacobian of this transformation.

Answer The Jacobian is $\det\begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix}=\det\begin{pmatrix}2u & -2v\\ 2v & 2u\end{pmatrix}=4u^2+4v^2$.

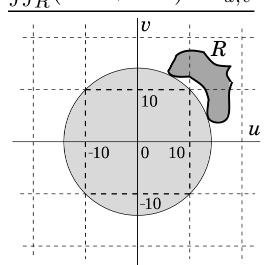
b) Suppose R is in the u,v plane and that Q is the image under F of R in the x,y plane. The area of Q is $\iint_Q 1\,dA_{x,y}$. Use the Change of Variables Formula to write an integral expression which is equal to this area using the u,v variables over R and the specific Jacobian computed in a).

Answer The Change of Variables Formula is

$$\iint_R(\text{Func described with } u \text{ and } v)\,|\textbf{JAC}|\,dA_{uv}=\iint_Q(\text{Func described with } x \text{ and } y)\,dA_{xy}$$

The Func is just 1 and **JAC** is what was computed in a). So the area of Q is equal to $\underline{\iint_R(4u^2+4v^2)\,dA_{u,v}}$.

c) A region R is shown in the u,v plane. Look carefully at the location of R compared to the coordinate axes. The region is mapped to a region Q in the x,y plane by the mapping whose Jacobian is computed in part a). If the area of R is at least 50, use complete English sentences to explain why the area of the image region Q is at least 40,000.



Hint No exact computation can be made since precise information isn't given. Underestimate the Jacobian for (u,v) in R using R's location. Combine this with b)'s answer to get the desired result.

Answer We work with the u,v integral underlined in b). The circle drawn is $u^2+v^2=2\cdot 10^2$. Since R is outside that circle, when (u,v) is in R, then $u^2+v^2\geq 2\cdot 10^2$. Therefore when (u,v) is in R, the Jacobian, $4u^2+4v^2$, is at least 800. The area of R is $\iint_R 1\,dA_{u,v}$ which is >50. If we integrate a function at least 800 over a region with area >50, the result must be $>50\cdot 800=40{,}000$. This is the minimum value of the integrand multiplied by the area of the domain.

Note Other reasonable explanations connecting the location of R and the size of u and v were certainly accepted.