Spring 2010

4. The 3-D parametric curve C $\begin{cases} x=\cos t \\ y=\sin t \\ z=t^2 \end{cases}$

a. Verify that the curvature is given by the formula $K(t)=\sqrt{\dfrac{5+4t^2}{(4t^2+1)^3}}$.

$$K(t)=\frac{\|r'(t)\times r''(t)\|}{\|r'(t)\|^3}$$

$$r(t)=\langle \cos t, \sin t, t^2\rangle$$

$$r'(t)=\langle -\sin t, \cos t, 2t\rangle \qquad r''(t)=\langle -\cos t, -\sin t, 2\rangle$$

$$r'(t)\times r''(t)=\begin{vmatrix} i & j & k \\ -\sin t & \cos t & 2t \\ -\cos t & -\sin t & 2 \end{vmatrix} = (2\cos t+2t\sin t)i-\underbrace{(-2\sin t+2t\cos t)j}_{+2\sin t-2t\cos t}+\underbrace{(\sin^2 t+\cos^2 t)}_{1}k$$

$$r'(t)\times r''(t)=\langle 2\cos t+2t\sin t, 2\sin t-2t\cos t, 1\rangle$$

$$\|r'(t)\times r''(t)\|=\sqrt{(2\cos t+2t\sin t)^2+(2\sin t-2t\cos t)^2+1^2}$$

$$=\sqrt{2^2(\cos t+t\sin t)^2+2^2(\sin t-t\cos t)^2+1}$$

$$=\sqrt{2^2(\cos^2 t+t^2\sin^2 t+2t\sin t\cos t)+2^2(\sin^2 t-2t\sin t\cos t+t^2\cos^2 t)+1}$$

$$\sqrt{4(\cos^2 t+t^2\sin^2 t+2t\sin t\cos t+\sin^2 t-2t\sin t\cos t+t^2\cos^2 t)+1}$$

$$=\sqrt{4(\underbrace{\cos^2 t+\sin^2 t}_{1}+t^2\underbrace{(\sin^2 t+\cos^2 t)}_{1})+1}=\sqrt{4(1+t^2)+1}=\sqrt{4+4t^2+1}=\sqrt{5+4t^2}$$

$$\|r'(t)\|=\|\langle -\sin t, \cos t, 2t\rangle\|=\sqrt{(-\sin t)^2+(\cos t)^2+(2t)^2}=\sqrt{\underbrace{\sin^2 t+\cos^2 t}_{1}+4t^2}=\sqrt{4t^2+1}$$

so $K(t)=\dfrac{\|r'(t)\times r''(t)\|}{\|r'(t)\|^3}=\sqrt{\dfrac{5+4t^2}{(4t^2+1)^3}}$

b. Explain why $\lim_{t\to\infty} K(t)=0$.

Since $K(t)=\sqrt{\dfrac{5+4t^2}{(4t^2+1)^3}}$, as $t\to\infty$, the denominator increases faster than the numerator increases because the power of the denominator is much higher than the power of the numerator, so as $t\to\infty$, $K(t)$ approaches 0.

c. The first 2 coordinates of this curve describe uniform circular motion. Explain why this statement is ~~consisted~~ consistent with the limit evaluated in b.