(12) 1. a) Suppose $\{v_1, v_2, \ldots, v_k\}$ is a set of vectors in $\mathbb{R}^n$. Define "$\{v_1, v_2, \ldots, v_k\}$ *is linearly independent*".

b) Verify that this set of three vectors in $\mathbb{R}^4$ is linearly independent: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$.

(18) 2. Here $A = \begin{bmatrix} 2 & -4 & -1 & 4 \\ -1 & 2 & -3 & -9 \\ 3 & -6 & -2 & 5 \end{bmatrix}$, which has RREF equal to $\begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

a) Give the dimension and an orthogonal basis of the null space of A.
b) Give the dimension and an orthogonal basis of the column space of A.

(16) 3. In this problem A is the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. The aim is to *diagonalize A orthogonally.* That is, we want to write $A = PDP^{-1}$ where D is a diagonal matrix and P is an orthogonal matrix.

a) Find the characteristic polynomial of A and find the two distinct eigenvalues of A.
b) Find an eigenvector for each of the eigenvalues listed in a). Check that these vectors are orthogonal.
c) Use the results of a) and b) to write D and P. D should be a diagonal matrix and P should be an orthogonal matrix.
d) Find P^{-1}. This should be *extremely easy* if P is orthogonal as was requested.
e) Verify PDP^{-1} is A: **first** compute PD; **then** multiply the result on the right by P^{-1}.

(6) 4. Explain why any 6 vectors in $\mathbb{R}^5$ must be linearly dependent.

(8) 5. Compute the determinant of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 10 & 9 & 8 & 7 \\ 6 & -5 & 2 & 7 \end{bmatrix}$.

(18) 6. For which values of c can the matrix $\begin{bmatrix} 3 & c \\ 2 & 1 \end{bmatrix}$ be diagonalized?

Comment Consider *all* values of c. There will be various special cases. Be sure you analyze each case, and supply a conclusion with supporting reasoning in each case.

(10) 7. **True or false** You must briefly justify your answers.
a) If A is a matrix for which the sum $A + A^T$ is defined, then A is a square matrix.
b) If A is a 2×2 matrix with distinct real eigenvalues, then the corresponding eigenvectors are always orthogonal.

(10) 8. Suppose A and B are 2×2 matrices with $A^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$.

a) Compute the inverse to AB^T.
b) Find a vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that $(BA)X = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

(12) 9. $\left\{ \begin{bmatrix} \frac{1}{3} \\ 0 \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} \right\}$ is an orthonormal basis of a subspace S of $\mathbb{R}^4$. Suppose $v = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -3 \end{bmatrix}$.

a) v can be written as a sum of v_1 in S and v_2 in $S^\perp$. Find v_1 and v_2.
b) What is the distance from v to S? You need <u>not</u> simplify your answer.
c) What is the dimension of $S^\perp$?

(10) 10. Suppose v and w are any vectors in $\mathbb{R}^n$. Prove that $\| v + w \|^2 + \| v - w \|^2 = 2 \| v \|^2 + 2 \| w \|^2$.
Hint Write everything in terms of dot products and *use linear algebra.* The equation given in this problem is called the *Parallelogram Law.*

(12) 11. In this problem, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$.

a) Find A^{-1}.

b) Use your answer to part a) to find a solution to the linear system $\begin{cases} 1x_1 + 1x_2 + 1x_3 = 2 \\ 1x_1 + 2x_2 + 3x_3 = 0 \\ 1x_1 + 4x_2 + 9x_3 = 1 \end{cases}$.

c) How many solutions does the linear system in b) have? Briefly justify your assertion.

(12) 12. Suppose A is a 5×5 matrix, and v is a non-zero vector in $\mathbb{R}^5$ so that $Av = 7v$.

a) Is $3v$ an eigenvector of A? If it is, what is the associated eigenvalue? Explain your answer.
b) Suppose additionally that A is invertible. Is v an eigenvector of A^{-1}? If it is, what is the associated eigenvalue? Explain your answer.

(10) 13. Suppose that A is a <u>symmetric</u> 3×3 matrix. Suppose also that v_1, v_2, and v_3 are <u>unit eigenvectors</u> of A with associated eigenvalues 5, -3, and 0 respectively. Further, suppose that the vector $w = 4v_1 + 6v_2 + 8v_3$.
a) What is Aw in terms of v_1, v_2, and v_3? Compute $\| Aw \|$.
b) If n is a positive integer, what is $A^n w$ in terms of v_1, v_2, and v_3?

(10) 14. Find the rank and nullity of the following matrix: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 0 & -4 & 1 \end{bmatrix}$.

(12) 15. a) Suppose v_1, v_2, $\ldots$, and v_k are vectors in $\mathbb{R}^n$ and also that w is a vector in $\mathbb{R}^n$. Define w *is a linear combination of* v_1, v_2, $\ldots$, *and* v_k.

b) Is $w = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ equal to a linear combination of $v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}$?

(12) 16. a) Suppose $A = \begin{bmatrix} 17 & -6 & 33 & 0 & 0 \\ 78 & 44 & 12 & 0 & -1 \\ 21 & -87 & 44 & 0 & 1 \\ -51 & 27 & 88 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 56 & 87 & -33 & 44 & 78 & 9 \\ 3 & 1 & 2 & 4 & 0 & -2 \end{bmatrix}$.

Compute AB if possible or indicate why this is not possible.
Compute BA if possible or indicate why this is not possible.
b) If C and D are 8×8 matrices, and D is *not* invertible, explain why C^2D^3 is not invertible.

(12) 17. All matrices in this problem will be 2×2. Specifically, $P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

a) Compute AP and PA. Show that if A and P commute (so $AP = PA$) then A must be a diagonal matrix.
b) Compute AQ and QA. Show that if A and Q commute (so $AQ = QA$) then the main diagonal entries of A must be equal *and* the entries of A on the "other" diagonal must be equal (all four entries don't have to be equal – the entries just must be equal in pairs).
c) Suppose that A commutes with both P and Q. Use the conclusions of a) and b) to show that such an A must be a scalar multiple of the identity matrix, I_2.
Comment P and Q resemble the *Pauli matrices* which are used in quantum mechanics.

Final Exam for Math 250, sections 1 & 5

May, 2011

Name ______________________________

Do all problems, in any order.
Show your work. An answer alone may not receive full credit.
No notes and no calculators may be used on this exam.
"Simplification" of answers is not necessary.

Problem Number	Possible Points	Points Earned:
1	12	
2	18	
3	16	
4	6	
5	8	
6	18	
7	10	
8	10	
9	12	
10	10	
11	12	
12	12	
13	10	
14	10	
15	12	
16	12	
17	12	
Total Points Earned:		