Here are answers to Version A. Other methods may also be correct.

(15) 1. This problem is about the following system of linear equations: [BELOW TO THE LEFT.]

$$\begin{cases} 3x_1 - 2x_2 + 2x_3 + 1x_4 = a \\ -6x_1 + 4x_2 + 4x_3 + 5x_4 = b \\ -15x_1 + 10x_2 + 6x_3 + 9x_4 = c \\ 18x_1 - 12x_2 - 4x_3 - 8x_4 = d \\ -9x_1 + 6x_2 + 2x_3 + 4x_4 = e \end{cases} \qquad \begin{bmatrix} 1 & -\frac{2}{3} & 0 & -\frac{1}{4} & -\frac{1}{12}b + \frac{1}{6}a \\ 0 & 0 & 1 & \frac{7}{8} & \frac{1}{8}b + \frac{1}{4}a \\ 0 & 0 & 0 & 0 & a - 2b + c \\ 0 & 0 & 0 & 0 & -2a + 2b + d \\ 0 & 0 & 0 & 0 & -b + a + e \end{bmatrix}$$

After some row operations, the augmented matrix of this system becomes what follows. [ABOVE TO THE RIGHT.] The following questions refer to the *coefficient matrix* of the system of linear equations. In this problem, you need not supply verification of your answers.

a) Give the dimension and a basis of the <u>null space</u> of the coefficient matrix.
Answer Here $a = b = c = d = e = 0$ so (looking at the row-reduced form) $x_1 = \frac{2}{3}x_2 + \frac{1}{4}x_4$ and $x_3 = -\frac{7}{8}x_4$. x_2 and x_4 are free variables. The dimension is 2 and a basis is $\left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} \\ 0 \\ -\frac{7}{8} \\ 1 \end{bmatrix} \right\}$.

b) Give the dimension and a basis of the <u>column space</u> of the coefficient matrix.
Answer The dimension is 2. The textbook's method is to use the column vectors in the original coefficient matrix corresponding to the pivots in the row-reduced form. Those vectors are the first and third, so a basis would be $\left\{ \begin{bmatrix} 3 \\ -6 \\ -15 \\ 18 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ -4 \\ 2 \end{bmatrix} \right\}$. The method discussed in class and used in some quiz solutions is this: a vector will be in the column space when $a - 2b + c = 0$, $-2a + 2b + d = 0$, and $-b + a + e = 0$. This means $c = -a + 2b$, $d = 2a - 2b$, and $e = -a + b$. The dimension will again be 2 (thank goodness!) and a basis can be assembled using a and b as free variables. It is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \right\}$. It is not too difficult (!) to see that the spans of the two pairs of vectors are identical.

c) Give the dimension and a basis of the <u>row space</u> of the coefficient matrix.
Answer The dimension of the row space is 2 and (following the textbook's notation: all vectors are column vectors!) a basis is $\left\{ \begin{bmatrix} 1 \\ -\frac{2}{3} \\ 0 \\ -\frac{1}{4} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{7}{8} \end{bmatrix} \right\}$. It is also true in this case (but *not always!* – the matrix could have two top rows of all 0's, for example!) that the transposes of the two top rows will be a basis: $\left\{ \begin{bmatrix} 3 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ 4 \\ 5 \end{bmatrix} \right\}$.

(10) 2. a) Suppose that S is a set of vectors in $\mathbb{R}^n$. Define "*S is a subspace of* $\mathbb{R}^n$".
Answer S is a subspace of $\mathbb{R}^n$ if the zero vector is in S, if when v_1 and v_2 are any vectors in S, then $v_1 + v_2$ must be in S, and if when v is any vector in S and c is any scalar, then cv is in S.

b) Suppose S is the set of vectors in $\mathbb{R}^2$ given by $\begin{bmatrix} x \\ y \end{bmatrix}$ where $y = x^2$ and x is any real number. Use your answer to part a) to explain why S is <u>not</u> a subspace of $\mathbb{R}^2$.

Answer The vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are both in this S (take $x = 1$ and $x = 2$ respectively in the description given). The sum of these vectors is $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$. This vector is *not* in S since if the first coordinate is 3, in order to be in this S the second coordinate must be $3^2 = 9 \neq 5$. So the second condition ("Closure under vector addition" as I called it in class one day) is not satisfied.

(15) 3. In this problem A is the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$. The aim is to *diagonalize* A. That is, we want to write $A = PDP^{-1}$ where D is a diagonal matrix and P is an invertible matrix.
a) Find the characteristic polynomial of A and find the two distinct eigenvalues of A.
Answer The characteristic polynomial is $\det \begin{bmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{bmatrix}$ which is $(1-\lambda)(-4-\lambda) - 2\cdot 3 = \lambda^2 + 3\lambda - 10$. But $\lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2)$ so that the eigenvalues are -5 and 2.
b) Find an eigenvector for each of the eigenvalues listed in a).
Answer $\underline{\lambda = -5}$ We want a non-zero vector in the nullspace of $A + 5I_2 = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$. A solution is apparent: $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. (Yes, you can do RREF etc, but for dimension 2 and small integer entries, I think an answer *is* apparent and this is enough supporting information.)
$\underline{\lambda = 2}$ We want a non-zero vector in the nullspace of $A - 2I_2 = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$. A solution is apparent: $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
c) Use the results of a) and b) to write D and P.
Answer $D = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$.
d) Find P^{-1}.
Answer Start with $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{array}\right]$ and use row ops to get I_2 on the left. Here is what I did (I think the row operations are fairly clear): $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 7 & 3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{7} & \frac{1}{7} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{7} & -\frac{2}{7} \\ 0 & 1 & \frac{3}{7} & \frac{1}{7} \end{array}\right]$ so that $P^{-1} = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{bmatrix}$. This is only 2×2 and other methods also work. I did check my answer by multiplying P with my candidate for P^{-1} (I got I_2).
e) Verify PDP^{-1} is A: **first** compute PD; **then** multiply the result on the right by P^{-1}.
Answer Let's start with $PD = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 15 & 2 \end{bmatrix}$ and then compute $\begin{bmatrix} -5 & 4 \\ 15 & 2 \end{bmatrix} P^{-1} =$ $\begin{bmatrix} -5 & 4 \\ 15 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{bmatrix} = \begin{bmatrix} -\frac{5}{7} + \frac{12}{7} & \frac{10}{7} + \frac{4}{7} \\ \frac{15}{7} + \frac{6}{7} & -\frac{30}{7} + \frac{2}{7} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = A$ so we're done.

(6) 4. Suppose S is a subspace of $\mathbb{R}^{500}$ which contains a set of 4 linearly independent vectors and which is spanned by 6 of its vectors. What are the possible values of the dimension of S? You *must* give reasoning which supports your answer to earn full credit.
Answer Any linearly independent set can be increased to a basis, and any spanning set can be "shrunk" to form a basis. Therefore a basis of S will have at least 4 vectors and at most 6 vectors. The dimension of S must be 4, 5, or 6.

(12) 5. a) Suppose $A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a 3×3 matrix with row vectors a, b, and c. Assume that $\det A = 5$. Find the determinant of the matrix $\begin{bmatrix} c+b \\ a+2b \\ a-b-c \end{bmatrix}$.

Answer The determinant of $\begin{bmatrix} c+b \\ a+2b \\ a-b-c \end{bmatrix}$ is unchanged if we subtract row 1 from row 3. The result is $\begin{bmatrix} c+b \\ a+2b \\ a \end{bmatrix}$. Now subtract row 3 from row 2, again not changing the determinant. The result is $\begin{bmatrix} c+b \\ 2b \\ a \end{bmatrix}$.

Now subtract half of row 2 from row 1, still not changing the determinant. The result is $\begin{bmatrix} c \\ 2b \\ a \end{bmatrix}$. If we "pull out" the 2 in the second row, the determinant will change by a factor of 2. So we will need to compute the determinant of $\begin{bmatrix} c \\ b \\ a \end{bmatrix}$ and double the result. But we may interchange row 1 and row 3 and only change the sign of the determinant. Then we can use the given determinant's value. So what we want is $(2)(-1)(5) = -10$.

b) Compute the determinant of the matrix $\begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & -1 & 1 & 2 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$.

Answer Expand along the first row. Then we want $3\det\begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} - 4\det\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$. But $\det\begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} = 2\det\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} = 2\left((-1)(-1) - 2(1)\right) = 2(-1) = -2$ (by expanding along the second row). I'll compute $\det\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ by expanding along the third row. So $\det\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} =$ $1\det\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} - 1\det\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = -4$. Put things together: $3(-2)-4(-4) = -6+16 = 10$, the value requested.

(12) 6. **True or false** (An answer alone will not receive full credit!)

a) If A is a 10×13 matrix, then the nullspace of A is *not* $\{0\}$.

Answer This is **TRUE**. The rank of A is at most 10 (the number of rows). Since the number of columns is 13, the nullity is at least $13-10=3$. There are at least 3 free variables, and the dimension of the nullspace is at least $3>0$: it is a subspace which has vectors other than the zero vector.

b) A basis of $\mathbb{R}^4$ is given by the vectors $\left\{ \begin{bmatrix} 1 \\ -3 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 8 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \\ -2 \end{bmatrix} \right\}$.

Answer This is **FALSE**. The dimension of $\mathbb{R}^4$ is 4. Therefore any basis of $\mathbb{R}^4$ must have 4 vectors, but 5 vectors are given here. Since $5\neq 4$ the statement is incorrect.

c) If A and B are any 2×2 matrices, then the matrix products AB and BA are equal.

Answer This is **FALSE**. If $A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $AB=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $BA=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. These are not equal.

(12) 7. Suppose A is the matrix $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ and B is the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. For which values of μ is $A - \mu B$ invertible? (Copy A and B carefully!)

Answer Compute $\det(A - \mu B)$. The matrix $A - \mu B$ will be invertible exactly when the determinant is not zero. Since $A - \mu B = \begin{bmatrix} 2-\mu & 1 & 2-\mu \\ 1 & 1 & -\mu \\ 0 & 2-\mu & 0 \end{bmatrix}$, the determinant can be computed easily by expanding along the third row. The value is $-(2-\mu)\det\begin{bmatrix} 2-\mu & 2-\mu \\ 1 & -\mu \end{bmatrix} = -(2-\mu)\big((2-\mu)(-\mu) - (2-\mu)1\big) = -(2-\mu)(2-\mu)(-\mu-1)$. The roots of this polynomial are 2 and -1. The matrix $A - \mu B$ is invertible for all other values of μ.

(18) 8. In this problem A is a 4×4 matrix, and v is an eigenvector of A with corresponding eigenvalue 3. You *must* give reasoning which supports your answer to earn full credit.

a) Is $-2v$ an eigenvector of A? If it is, what is its corresponding eigenvalue?

Answer YES Just consider $A(-2v) = -2Av = (-2)3v = 3(-2v)$. Therefore $-2v$ is an eigenvector of A and its corresponding eigenvalue is $\lambda = 3$.

b) Is v an eigenvector of $5A$? If it is, what is its corresponding eigenvalue?

Answer YES Just consider $(5A)v = 5(Av) = 5(3v) = (15)v$. Therefore v is an eigenvector of $5A$ and its corresponding eigenvalue is $\lambda = 15$.

c) Is v an eigenvector of A^2? If it is, what is its corresponding eigenvalue?

Answer YES Just consider $A^2v = (AA)v = A(Av) = A(3v) = 3(Av) = 3(3v) = 9v$. Therefore v is an eigenvector of A^2 and its corresponding eigenvalue is $\lambda = 9$.