640:250:1 & 5 **Answers to the First Exam** **3/3/2011**

Here are answers to Version A. Other methods may also be correct.

(16) 1. This problem discusses the following system of linear equations: $\begin{cases} 5x_1 + x_2 + x_3 - 2x_4 = a \\ -3x_1 + 4x_2 \quad + x_4 = b \\ 7x_1 + 6x_2 + 2x_3 - 3x_4 = c \end{cases}$.

Note We use **GOAT** to answer this question.

a) Find specific numbers for which this system has <u>*no solution*</u>.

Answer There is *no solution* if $c + 2a - b \neq 0$ because the last row of **GOAT**'s right side represents the equation $0x_1 + 0x_2 + 0x_3 + 0x_4 = c + 2a - b$. An example with specific numbers is $a = 0$, $b = 0$, and $c = 1$.

b) Find specific numbers (*not all 0!*) for which this system has a solution, and display one specific solution.

Answer If $c = 1$, $a = 0$, and $b = 1$ then the last row is $0 = 0$. The other rows of **GOAT** (when $x_3 = 0$ and $x_4 = 0$) imply that $x_1 = -\frac{1}{23}$ and $x_2 = \frac{5}{23}$.

c) Consider the collection of all possible $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ for which this system has a solution. Write the set of such vectors as a span.

Answer Here the condition is $c - 2a - b = 0$ so $c = 2a + b$. Then $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ 2a+b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ so that the vectors we want to describe are the span of $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

d) Consider the collection of all possible $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ which are solutions for the associated homogeneous system. Write the set of such vectors as a span.

Answer If all of a, b, and c are 0, then the last row of **GOAT** has no content. The first two rows tell us that x_3 and x_4 are free variables, and that $x_1 = -\frac{4}{23}x_3 + \frac{9}{23}x_4$ and $x_2 = -\frac{3}{23}x_3 + \frac{1}{23}x_4$. Therefore $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{4}{23}x_3 + \frac{9}{23}x_4 \\ -\frac{3}{23}x_3 + \frac{1}{23}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -\frac{4}{23} \\ -\frac{3}{23} \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} \frac{9}{23} \\ \frac{1}{23} \\ 0 \\ 1 \end{bmatrix}$ so that the set of solutions of the associated homogeneous system is the span of $\begin{bmatrix} -\frac{4}{23} \\ -\frac{3}{23} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{9}{23} \\ \frac{1}{23} \\ 0 \\ 1 \end{bmatrix}$.

(10) 2. Consider these four vectors in $\mathbb{R}^4$: $v_1 = \begin{bmatrix} 2 \\ 11 \\ -6 \\ 7 \end{bmatrix}$, $v_2 = \begin{bmatrix} 8 \\ -12 \\ 18 \\ 14 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}$, and $v_4 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 5 \end{bmatrix}$.

Is the set $\{v_1, v_2, v_3, v_4\}$ linearly dependent or linearly independent? If it is linearly dependent, find a non-trivial linear combination which is equal to the zero vector. If it is linearly independent, explain why.

Note We use **ELEPHANT** to answer this question.

Answer Since **ELEPHANT** has nullity$=2$ there will be non-trivial solutions to $\sum_{j=1}^{4} c_j v_j = 0$. Take $c_3 = 14$ and $c_4 = 0$. Then $c_1 = -2$ and $c_2 = -3$ provide a solution. Therefore we know that $-2v_1 - 3v_2 + 14v_3 = 0$, a non-trivial linear combination equal to the zero vector.

(8) 3. Find the rank and the nullity of the 5×5 matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 6 & 8 & 10 \\ 4 & 7 & 10 & 13 & 16 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$. **Answer** The row operations

$\begin{cases} r_2-3r_1\to r_2 \\ r_3-2r_1\to r_3 \\ r_4-4r_1\to r_4 \end{cases}$ result in the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -2 & -4 & -6 & -8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$. Then $\begin{cases} -\frac{1}{2}r_2\to r_2 \\ r_4-r_2\to r_4 \end{cases}$ give $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$. Do

$\begin{cases} r_3\leftrightarrow r_5 \\ r_1-r_2\to r_1 \end{cases}$ to get $\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Finally, $\begin{cases} r_1-r_3\to r_1 \\ r_2-2r_3\to r_2 \end{cases}$ produce $\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. This is in RREF.
The rank is 3 and the nullity is 2.

(10) 4. a) Find a set of 4 linearly independent vectors in $\mathbb{R}^5$. You must give evidence that this set is linearly independent.

Answer The first 4 standard vectors can be used: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. If $\sum_{j=1}^{4} c_j v_j = 0$ where v_j is the j^{th} vector in the preceding list, then the j^{th} component $(1 \le j \le 4)$ shows that $c_j = 0$. There is no non-trivial linear combination which equals the zero vector so this set is linearly independent.

b) Explain why any 4 vectors in $\mathbb{R}^3$ must be linearly dependent.

Answer Suppose $\{v_1, v_2, v_3, v_4\}$ is any set of 4 vectors in $\mathbb{R}^3$. The coefficient matrix of the linear system $\sum_{j=1}^{4} c_j v_j = 0$ can be put into RREF. The result is a 3×4 matrix which has at most 3 pivots, and its nullity is at least $4-3=1$. There will always be a free variable (at least 1). Any free variables can be given non-zero values, and then there will be a non-trivial solution to the original linear system.

(8) 5. Suppose A and B are 2×2 matrices with $A^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$.

a) Compute the inverse to AB^T.

Answer $\left(AB^T\right)^{-1} = \left(B^{-1}\right)^T A^{-1}$ so we compute $\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ which is $\begin{bmatrix} -1 & 13 \\ 3 & -4 \end{bmatrix}$.

b) Find a vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that $(BA)X = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Answer $(BA)X = B(AX)$ so $AX = B^{-1}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -12 \end{bmatrix}$. Then $X = A^{-1}\begin{bmatrix} -1 \\ -12 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ -12 \end{bmatrix} = \begin{bmatrix} -23 \\ -35 \end{bmatrix}$.

(4) 6. Give an example of a matrix M in <u>reduced row echelon form</u> so that the rank of M is **3**, the nullity of M is **2**, and the nullity of M^T is **1**. You need *not* verify your example.

Answer $M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(12) 7. a) Find the inverse of the matrix $W = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$. **Answer** Start with $\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$.
Use row operations so that I_4 will appear on the left.

$$\downarrow \begin{cases} \frac{1}{2}r_2 \to r_2 \\ -\frac{1}{2}r_3 \to r_3 \\ r_4 - r_1 \to r_4 \end{cases} \qquad \downarrow \begin{cases} r_1 - 2r_3 \to r_1 \\ r_4 + 2r_3 \to r_4 \end{cases} \qquad \downarrow -\tfrac{1}{3}r_3 \to r_3$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & -3 & -1 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -3 & -1 & 0 & -1 & 1 \end{array}\right] \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

Finally, $\begin{cases} r_1 - 4r_4 \to r_1 \\ r_2 - \frac{3}{2}r_4 \to r_2 \end{cases}$ gives $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & \frac{4}{3} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} \end{array}\right]$ so W^{-1} is $\begin{bmatrix} -\frac{1}{3} & 0 & -\frac{1}{3} & \frac{4}{3} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$.

b) Check your result: directly multiply W by your candidate for W^{-1}.

Answer Indeed, truly: $\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & 0 & -\frac{1}{3} & \frac{4}{3} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(10) 8. For which values of w is the span of the vectors $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $v_4 = \begin{bmatrix} w+2 \\ w \\ 2 \\ 5 \end{bmatrix}$ equal to ***all*** of $\mathbb{R}^4$? That is, given *any* vector $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$, find all values of w so that some linear combination of $\{v_1, v_2, v_3, v_4\}$ will equal B. Support your assertion with reasoning explained in complete English sentences. **Note** We use **BEAR** to answer this question.

Answer We need to know if we can *always* solve $\sum_{j=1}^{4} c_j v_j = B$. **BEAR** changes this system to an equivalent system. If $17 + 6w \neq 0$, then the changed system can always be solved by back substitution, getting c_4 from the lowest row, then c_3 (using c_4) from the third row, etc. If $17 + 6w = 0$ and we take, for example, $b_4 = 1$, $b_1 = 0$, $b_2 = 0$, and $b_3 = 0$, then the last equation becomes $0 = 1$ and there is no solution. Therefore the span is all of $\mathbb{R}^4$ exactly when $w \neq -\frac{17}{6}$.

(8) 9. **True or false** You must briefly justify your answers.

a) If A is a matrix for which the sum $A + A^T$ is defined, then A is a square matrix. **Answer** This is **TRUE**. If A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix. The sum is defined when the dimensions agree, so that m must equal n. Therefore A is square.

b) If A and B are any 2×2 matrices, then the matrix products AB and BA are equal. **Answer** This is **FALSE**. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. These aren't equal.

(6) 10. a) Suppose v_1, v_2, …, and v_k are vectors in $\mathbb{R}^n$ and also that w is a vector in $\mathbb{R}^n$. Define "w is a linear combination of v_1, v_2, …, and v_k" using complete English sentences. **Answer** w is a linear combination of v_1, v_2, …, and v_k if there are scalars c_1, c_2, …, and c_k so that $\sum_{j=1}^{k} c_j v_j = w$.

b) Give an example of a vector w in $\mathbb{R}^3$ and vectors v_1 and v_2 in $\mathbb{R}^3$ so that w is *not* a linear combination of v_1 and v_2. Give evidence supporting your assertion. **Answer** If $w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and both v_1 and v_2 are the zero vector, then any linear combination of v_1 and v_2 is the zero vector. But $w \neq 0$.

(8) 11. Let u be a solution of $Ax = b$ and v be a solution of $Ax = 0$, where A is an $m \times n$ matrix and b is a vector in $\mathbb{R}^m$. Show that $u + v$ is a solution of $Ax = b$.

Answer $A(u + v) = Au + Av = b + 0 = b$.