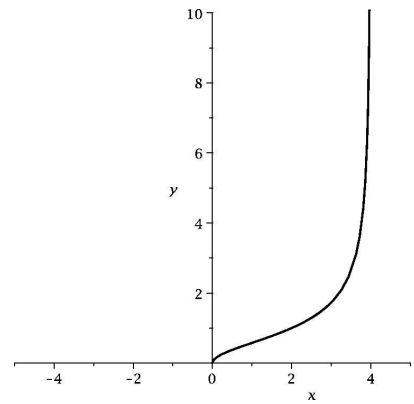
Here is a correct answer to the first workshop problem. On the next page are some comments about this solution. We hope that these general comments and the suggestions written on your assignment will help you with your future writeups.

2. Suppose $f(x) = \sqrt{\frac{x}{4-x}}$.

a) Find the graph of $y = f(x)$ in the window $-5 \le x \le 5$ and $0 \le y \le 10$.

Answer The graph, obtained using a machine, is shown to the right.



b) What is the domain of f? Verify your statement algebraically.

Answer The domain is $[0, 4)$, that is, all numbers x satisfying $0 \le x < 4$. In the expression $\sqrt{\frac{x}{4-x}}$ we must avoid dividing by 0 and avoid taking square roots of negative numbers. The first restriction means that x must not be 4. The second restriction is $\frac{x}{4-x} \ge 0$. This inequality can be true in two ways. First, we can require that $x \ge 0$ and $4 - x \ge 0$, which means $x \ge 0$ and $4 \ge x$. That is, $0 \le x \le 4$, but we may not have $x = 4$ to avoid division by 0, so certainly the interval $[0, 4)$ is in the domain. But $\frac{x}{4-x} \ge 0$ can also occur if $x \le 0$ and $4 - x \le 0$, which means $x \le 0$ and $4 \le x$. There are no numbers satisfying this pair of inequalities, so that $[0, 4)$ is all of the domain of f.

c) Solve $y = f(x)$ for x. What is the range of f? Your expression for x in terms of y may help to verify your statement algebraically.

Answer The range of f is $[0, \infty)$. First, the "outputs" from the formula $\sqrt{\frac{x}{4-x}}$ must always be non-negative, so that the range of f is at most $[0, \infty)$. If $y = \sqrt{\frac{x}{4-x}}$ we see $y^2 = \frac{x}{4-x}$ so that $4y^2 - xy^2 = x$ and $4y^2 = x + xy^2$, and then $4y^2 = x(1 + y^2)$. Therefore $x = \frac{4y^2}{1+y^2}$. Certainly this x is non-negative (all of the parts of the right-hand side of the previous equation are non-negative). Also, the x produced is less than 4, since the inequality $\frac{4y^2}{1+y^2} < 4$ is equivalent to $4y^2 < 4(1 + y^2)$ which is certainly correct since $0 < 4$. Therefore if y is any non-negative number, another number x can be computed which is in the interval $[0, 4)$ and which satisfies the equation $y = f(x)$. We have verified that the range of f is all of $[0, \infty)$.

See the other side for comments on this solution.

Comments

Correct spelling and neat presentation are essential to good writeups. Also, please staple multiple pages. A good solution will answer the questions which are asked. The answers should be direct and clear. What is written on the previous page is actually *much less* than what was written by many students, but the student solutions frequently did not answer the questions. Also, many student solutions were not complete or correct. We hope that the solution provided will help you learn what we're looking for!

Part a) The expectation for the graph is that students would use a graphing calculator (which is required for this course) and that the graph would be copied from the viewscreen of such an instrument. No specific function evaluation is needed for this part.

Part b) A specific answer for the domain must be given, and then algebraic reasoning must be used (it is *requested!*) to verify the asserted domain. Here such reasoning is characterized by manipulation of inequalities. No list of specific failed and successful function evaluations, with or without numerical evidence, is sufficient to verify the correct domain. There are infinitely many numbers in the domain and there are infinitely many numbers which are not in the domain so students will not have the time and paper to write enough lists of successful or failed evaluations to verify a domain

Part c) Several specific items are requested here. First, we request solution of the equation $y = f(x)$ for x. Such a solution must be presented, and some evidence of the process (which isn't completely routine, since a number of incorrect solutions were given) should be given. Second, an explicit range of f should be given. Then some verification of the asserted domain should be given. The square root symbol in this course and *everywhere* will always mean the non-negative square root. The function inverse to f, whose definition is given by the formula $\frac{4y^2}{1+y^2}$, has a "natural domain" which is all real numbers. But this is not the range of the original function f. The situation is somewhat complicated, and some discussion must be given to verify that the suggested range is actually correct.