**Solutions for 151 Televised Final Review**

1) Find

$$\lim_{x\to 0} x^x$$

This is an indeterminate form because it is of the form $0^1$. We let $y = x^x$, so $\ln y = x \ln x$.

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln x}{1/x} = \lim_{x\to 0} \frac{1/x}{-1/x^2} = \lim_{x\to 0} -x = 0$$

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{\ln y} = e^0 = 1$$

We can apply L'Hopital's rule because the limit was of the form infinity over infinity.

2) Find $f'(x)$ where

$$f(x) = (\cos x)^x$$

We let $y = (\cos x)^x$. Then $\ln y = x(\ln(\cos x))$. We take the derivative of both sides to find:

$$\frac{1}{y} y' = x \frac{1}{\cos x}(-\sin x) + \ln(\cos x)$$

Multiplying both sides by $y = (\cos x)^x$, we have

$$y' = (\cos x)^x \left(-x \tan x + \ln(\cos x)\right)$$

3) Approximate $(8.1)^{1/3}$ using linearization.

We let $f(x) = x^{1/3}$. We note that 8.1 is near 8, so we let $a = 8$. Then $f(a) = 8^{1/3} = 2$, and $f'(x) = (1/3)x^{-2/3}$, so $f'(a) = 1/12$. Then

$$(8.1)^{1/3} \approx f(a) + f'(a)(x-a) = 2 + \frac{1}{12}(8.1 - 8) = 2 + \frac{1}{12}(0.1)$$

4) Find $L_4$ and $R_4$ for $f(x) = x^2$ on the interval $[1, 3]$.

We have $\Delta x = (3-1)/4 = 1/2$. Then

$$L_4 = \frac{1}{2}\left[f(1) + f(3/2) + f(2) + f(5/2)\right] = \frac{1}{2}\left[1 + \frac{9}{4} + 4 + \frac{25}{4}\right] = \frac{1}{2}\left[5 + \frac{17}{2}\right] = \frac{27}{4}$$

$$R_4 = \frac{1}{2}\left[f(3/2) + f(2) + f(5/2) + f(3)\right]\frac{1}{2}\left[\frac{9}{4} + 4 + \frac{25}{4} + 9\right] = \frac{1}{2}\left[13 + \frac{17}{2}\right] = \frac{43}{4}$$

5) Two circles have the same center. The inner circle has radius $r$ which is increasing at $3in/s$. The outer circle has radius $R$ which is increasing at $2in/s$. Suppose that $A$ is the area of the region between the circles. When $r = 7in$ and $R = 10in$, what is $A$? How fast is $A$ changing? Is $A$ increasing or decreasing?

The area of the outer circle is $\pi R^2$ and the area of the inner circle is $\pi r^2$. Then $A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$. Then when $r = 7$ and $R = 10$, $A =$

$\pi(10^2 - 7^2) = 51\pi$. To find how fast $A$ is changing, we take the derivative with respect to time.

$$\frac{dA}{dt} = \pi\left(2R\frac{dR}{dt} - 2r\frac{dr}{dt}\right) = \pi(2 \cdot 10 \cdot 2 - 2 \cdot 7 \cdot 3) = \pi(-2)in^2/s$$

Since $dA/dt$ is negative, $A$ is decreasing.

6) Find $\dfrac{dy}{dx}$ if $y^3 = 3xy + y^2 + 4x^3$.

$$\begin{aligned}
3y^2\frac{dy}{dx} &= 3x\frac{dy}{dx} + 3y + 2y\frac{dy}{dx} + 12x^2 \\
\frac{dy}{dx}(3y^2 - 3x - 2y) &= 3y + 12x^2 \\
\frac{dy}{dx} &= \frac{3y + 12x^2}{3y^2 - 3x - 2y}
\end{aligned}$$

7) Find $\dfrac{dy}{dx}$ if $y = (\ln(x^2) + 2x^3)^{1/3}$.

$$\frac{dy}{dx} = \frac{1}{3}\left(\ln(x^2) + 2x^3\right)^{-2/3}\left(\frac{1}{x^2}(2x) + 6x^2\right) = \frac{1}{3}\left(\ln(x^2) + 2x^3\right)^{-2/3}\left(\frac{2}{x} + 6x^2\right)$$

8) Calculate

$$\lim_{x\to 0}\frac{\sin x - x}{e^x - 1}$$

This is of the form $0/0$, so we can apply L'Hopital.

$$\lim_{x\to 0}\frac{\sin x - x}{e^x - 1} = \lim_{x\to 0}\frac{\cos x - 1}{e^x} = \frac{\cos 0 - 1}{e^0} = \frac{0}{1} = 0$$

9) Calculate the following limits:

$$\lim_{x\to 3}\frac{x^2 + 5x + 6}{x^2 + 2x - 3}, \quad \lim_{x\to -3}\frac{x^2 + 5x + 6}{x^2 + 2x - 3}, \quad \lim_{x\to\infty}\frac{x^2 + 5x + 6}{x^2 + 2x - 3}$$

$$\lim_{x\to 3}\frac{x^2 + 5x + 6}{x^2 + 2x - 3} = \frac{9 + 15 + 6}{9 + 6 - 3} = \frac{30}{12} = \frac{5}{2}$$

The second limit is of the form $0/0$ so we can apply L'Hopital.

$$\lim_{x\to -3}\frac{x^2 + 5x + 6}{x^2 + 2x - 3} = \lim_{x\to -3}\frac{2x + 5}{2x + 2} = \frac{-1}{-4} = \frac{1}{4}$$

The third limit is of the form infinity over infinity. We can apply L'Hopital or we can use algebra. Algebraically,

$$\lim_{x\to\infty}\frac{x^2 + 5x + 6}{x^2 + 2x - 3} = \lim_{x\to\infty}\frac{1 + 5/x + 6/x^2}{1 + 2/x - 3/x^2} = 1$$

Using L'Hopital,

$$\lim_{x\to\infty}\frac{x^2+5x+6}{x^2+2x-3}=\lim_{x\to\infty}\frac{2x+5}{2x+2}=\lim_{x\to\infty}\frac{2}{2}=1$$

10) Let $f(x)=\cos(3x)+3x^2-6x+1$. Explain why there must be a root of $f(x)=0$ in $[-1,1]$.

First, we note that the function $f(x)$ is continuous on $[-1,1]$. Also, $f(-1)=\cos(-3)+3+6+1=10+\cos(-3)\geq 9>0$ because $-1\leq\cos x\leq 1$ for all $x$. Also, $f(1)=\cos(3)+3-6+1=-2+\cos(3)\leq -1<0$. Then, by the intermediate value theorem, there is $c$ in the interval $[-1,1]$ such that $f(c)=0$.

11) Suppose that the function $f$ is continuous on $[1,3]$ and differentiable on $(1,3)$. If $f(1)=9$ and $f'(x)\geq 2$ for $1\leq x\leq 3$, how small can $f(3)$ possibly be?

Since the function is continuous and differentiable, we can apply the mean value theorem. By the MVT, there is $c$ in $(1,3)$ such that

$$f'(c)=\frac{f(3)-f(1)}{3-1}=\frac{f(3)-9}{2}$$

. We also have $f'(c)\geq 2$ by assumption, so $(f(3)-9)/2\geq 2$, so $f(3)-9\geq 4$ and $f(3)\geq 13$.

12) Find the value of $c$ that makes the function continuous.

$$f(x)=\begin{cases} x^2-c & \text{for } x<5\\ 4x+2c & \text{for } x\geq 5\end{cases}$$

For the function to be continuous, we need

$$\lim_{x\to 5^+}f(x)=f(5)=\lim_{x\to 5^-}f(x)$$

We have

$$\lim_{x\to 5^+}f(x)=\lim_{x\to 5^+}4x+2c=20+2c$$

$$f(5)=20+2c$$

$$\lim_{x\to 5^-}f(x)=\lim_{x\to 5^-}x^2-c=25-c$$

Therefore, we have $25-c=20+2c$ so $3c=5$ and $c=5/3$.

13) Describe the set $S=\{x\in\mathbb{R}:|4x-3|>2 \text{ and } |x-2|\leq 1\}$ in terms of intervals.

The inequality $|4x-3|>2$ tells us that either $4x-3>2$ or $4x-3<-2$, so either $x>5/4$ or $x<1/4$.

The inequality $|x-2|\leq 1$ tells us that $-1\leq x-2\leq 1$, so $1\leq x\leq 3$.

If $x<1/4$ for the first inequality, the second inequality is not satisfied. Therefore, we need $x>5/4$. Then we already have $x\geq 1$ and the only extra condition is $x\leq 3$. Therefore, $S=(5/4,3]$.

14) Complete the square for $4x^2-6x+1$ to solve $4x^2-6x+1=0$.

We first factor out 4.

$$4x^2-6x+1 = 4\left(x^2-\frac{3}{2}x+\frac{1}{4}\right) = 4\left[\left(x-\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2+\frac{1}{4}\right] = 4\left[\left(x-\frac{3}{4}\right)^2-\frac{5}{16}\right]$$

Setting this to zero, we have $(x-(3/4))^2 = 5/16$, so $x-3/4 = \pm\sqrt{5}/4$, so $x = 3/4 \pm \sqrt{5}/4$.

15) Find the horizontal and vertical asymptotes of

$$\frac{2x}{\sqrt{3x^2-1}}$$

To find the vertical asymptotes, we set the denominator equal to zero, so we have vertical asymptotes, $x = \sqrt{1/3}$ and $x = -\sqrt{1/3}$.

To find the horizontal asymptotes, we take the limits.

$$\lim_{x\to-\infty}\frac{2x}{\sqrt{3x^2-1}} = \lim_{x\to-\infty}\frac{2x}{|x|\sqrt{3-\frac{1}{x^2}}} = \lim_{x\to-\infty}\frac{2x}{-x\sqrt{3-\frac{1}{x^2}}} = \lim_{x\to-\infty}\frac{-2}{\sqrt{3-\frac{1}{x^2}}} = \frac{-2}{\sqrt{3}}$$

$$\lim_{x\to\infty}\frac{2x}{\sqrt{3x^2-1}} = \lim_{x\to\infty}\frac{2x}{|x|\sqrt{3-1/x^2}} = \lim_{x\to\infty}\frac{2x}{x\sqrt{3-1/x^2}} = \lim_{x\to\infty}\frac{2}{\sqrt{3-1/x^2}} = \frac{2}{\sqrt{3}}$$

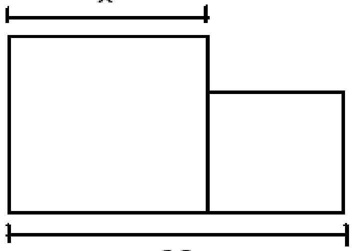
Therefore, the horizontal asymptotes are $y = -2/\sqrt{3}$ and $y = 2/\sqrt{3}$.

16) Find the equation of the tangent line to the function $f(x) = x^3+5x+2$ at $a = 3$.

$f'(x) = 3x^2+5$. The slope of the tangent line is $f'(a) = 27+5 = 32$. The tangent line also goes through the point $(a, f(a))$ and $f(a) = 27+15+2 = 44$, so the equation of the tangent line is

$$y-44 = 32(x-3)$$

17) Two squares are placed so their sides are touching as shown. The sum of the lengths of one side of each square is $10ft$. Suppose the length of the left square is $x$ feet. The left square is painted with paint costing \$6 per square foot. The right square is painted with paint costing \$4 per square foot. For which $x$ will the cost be a minimum?



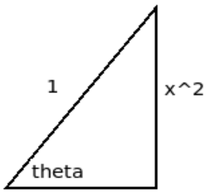

The side of the right square is $10-x$, so the cost is

$$C(x) = 6x^2+4(10-x)^2$$

To find the minimum, we take the derivative. $C'(x) = 12x + 8(10 - x)(-1) = 20x - 80 = 0$, so $x = 4$ is a critical point. Looking at the derivative, we see that $C(x)$ is decreasing to the left of 4 and increasing to the right, so $x = 4$ gives the minimum.

18) Simplify $\cos(\sin^{-1}(x^2))$.

There are two ways to approach this problem. Geometrically, we let $\theta = \sin^{-1}(x^2)$, so $\sin(\theta) = x^2$ and we draw a triangle.




The remaining side has length $\sqrt{1^2 - (x^2)^2} = \sqrt{1 - x^4}$. Therefore,

$$\cos(\sin^{-1}(x^2)) = \cos(\theta) = \sqrt{1 - x^4}$$

Alternatively, we can use trig identities.

$$\cos(\sin^{-1}(x^2)) = \sqrt{1 - \sin^2(\sin^{-1}(x^2))} = \sqrt{1 - (x^2)^2} = \sqrt{1 - x^4}$$

19) Solve $\ln(x^2 + 10) - \ln(x^2 + 1) = 3\ln 2$

We use properties of logarithms: $\ln(x^2 + 10) - \ln(x^2 + 1) = \ln((x^2 + 10)/(x^2 + 1))$ and $3\ln 2 = \ln(2^3) = \ln 8$. Then

$$\ln \frac{x^2 + 10}{x^2 + 1} = \ln 8 \Rightarrow \frac{x^2 + 10}{x^2 + 1} = 8 \Rightarrow x^2 + 10 = 8x^2 + 8 \Rightarrow 7x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2/7}$$

20) Use the limit definition of the derivative to calculate the derivative of $f(t) = t^{-2}$ at $a = 1$.

$$f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{\frac{1}{(1+h)^2} - 1}{h} = \lim_{h \to 0} \frac{-2h - h^2}{h(1 + h)^2} = \lim_{h \to 0} \frac{-2 - h}{(1 + h)^2} = -2$$

21) Find all functions $f(x)$ such that $f''(x) = x^2(x^3 + 2x + 1)$.

We have $f''(x) = x^5 + 2x^3 + x^2$ so

$$f'(x) = \frac{x^6}{6} + \frac{x^4}{2} + \frac{x^3}{3} + c$$

$$f(x) = \frac{x^7}{42} + \frac{x^5}{10} + \frac{x^4}{12} + cx + d$$

Note: you can use any letters you wish for the constants.

22) Differentiate $\tan^{-1}(\ln x)$, $\ln(\sin^{-1}(x^2))$, $(x^2+3)/(2x^2+1)$ and $\sin^3(e^{2x} + \ln x)$.

$$\frac{d}{dx}\tan^{-1}(\ln x) = \frac{1}{1+(\ln x)^2}\frac{1}{x}$$

$$\frac{d}{dx}\ln(\sin^{-1}(x^2)) = \frac{1}{\sin^{-1}(x^2)}\frac{1}{\sqrt{1-x^4}}(2x)$$

$$\frac{d}{dx}\frac{x^2+3}{2x^2+1} = \frac{(2x^2+1)(2x)-(x^2+3)(4x)}{(2x^2+1)^2}$$

$$\frac{d}{dx}\sin^3(e^{2x}+\ln x) = 3\sin^2(e^{2x}+\ln x)\left(\cos(e^{2x}+\ln x)\right)\left(2e^{2x}+\frac{1}{x}\right)$$

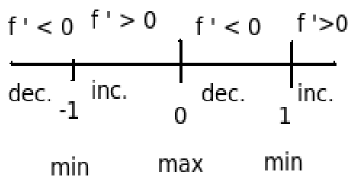
23) Let $f(x) = (1-x^2)^4$. Find the intervals where $f(x)$ is increasing, decreasing, concave up, and concave down. Give any inflection points and local extremas.

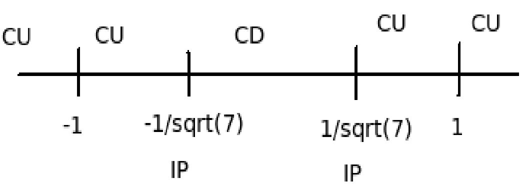
$$f'(x) = 4(1-x^2)^3(-2x) = -8x(1-x^2)^3$$

so the critical points are $-1, 0, 1$.

$$\begin{aligned} f''(x) &= -8x\cdot 3(1-x^2)^2(-2x)+(1-x^2)^3(-8) = 8(1-x^2)^2\left(6x^2-1+x^2\right) \\ &= 8(1-x^2)^2(7x^2-1) \end{aligned}$$

so the second derivative is zero at $-1, -\sqrt{1/7}, \sqrt{1/7}, 1$.




24) Find the maximum and minimum values of $f(x) = (\cos x)(\sin x)$ on the interval $[0, \frac{\pi}{2}]$.

$$f'(x) = (\cos x)(\cos x)+(\sin x)(-\sin x) = \cos^2 x-\sin^2 x = 1-\sin^2 x-\sin^2 x = 1-2\sin^2 x$$

Setting $f'(x) = 0$, we have $\sin x = \pm 1/\sqrt{2}$, In the interval, $[0, \pi/2]$, this means $x = \pi/4$. We now check the values of $f(x)$ at the endpoints and the critical point. $f(0) = 0$, $f(\pi/4) = 1/2$ and $f(\pi/2) = 0$, so the maximum is $1/2$ attained when $x = \pi/4$ and the minimum is 0 attained at both $x = 0$ and $x = \pi/2$.

25) Let $f(x) = \dfrac{1}{x+1}$. Give the intervals on which $f$ is increasing, decreasing, concave up, concave down. Give any asymptotes, inflection points and local extremas. Use this information to sketch the graph of $f(x)$.

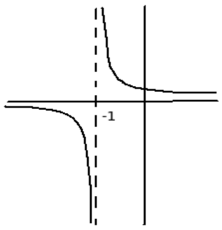
We write $f(x) = (x+1)^{-1}$, so $f'(x) = -(x+1)^{-2}$ and $f''(x) = 2(x+1)^{-3}$. We have a vertical asymptote at $x = -1$.

$$\lim_{x\to -1^+} f(x) = \infty, \quad \lim_{x\to -1^-} f(x) = -\infty$$

We also have

$$\lim_{x\to\infty} f(x) = 0 = \lim_{x\to -\infty} f(x) = 0.$$

Therefore, there is a horizontal asymptote at $y = 0$. The derivative is never zero but it is undefined at $-1$. The derivative is negative on both sides of $-1$, so there are no local extremum. The second derivative is positive when $x > -1$ and is negative when $x < -1$. Therefore, the function is concave up for $x > -1$ and concave down for $x < -1$, so there is an inflection point at $x = -1$. Here is a general sketch of the graph.




26) Use two iterations of Newton's method to approximate $\sqrt{5}$.

We note that $\sqrt{5}$ is a zero of $f(x) = x^2 - 5$. A good initial guess is $x_0 = 2$, since $\sqrt{4} = 2$. Then $f(x_0) = 2^2 - 5 = -1$. We have $f'(x) = 2x$, so $f'(x_0) = 4$. Then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{-1}{4} = \frac{9}{4}.$$

For the second interation, we have $f(x_1) = (9/4)^2 - 5 = 81/16 - 80/16 = 1/16$ and $f'(x_1) = 9/2$, so

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{9}{4} - \frac{1/16}{9/2} = \frac{9}{4} - \frac{1}{8\cdot 9} = \frac{9\cdot 18}{72} - \frac{1}{72} = \frac{162-1}{72} = \frac{161}{72}$$

Therefore, $\sqrt{5} \approx 161/72$.

27) Suppose $f(x)$ is a differentiable function with $f(9) = 3$, $f'(9) = 6$ and $f''(9) = -2$. If $F(x) = f(x^2)$, compute $F(3)$, $F'(3)$ and $F''(3)$.

$F(3) = f(3^2) = f(9) = 3$. We have $F'(x) = f'(x^2)(2x)$, so $F'(3) = f'(9)(6) = 36$. Then $F''(x) = f'(x^2)(2) + (2x)f''(x^2)(2x)$, so $F''(3) = 2f'(9) + 36f''(9) = 12 - 72 = -60$.

28) Find constants $a, b$ such that the function $f(x)$, defined below, is differentiable.

$$f(x) = \begin{cases} ax+1 & \text{if } x < 1 \\ x^2 + b & \text{if } x \geq 1 \end{cases}$$

To be differentiable, the function must be continuous, so we need $a(1)+1 = a+1 = 1^2 + b = b+1$, so $a = b$. The derivative of the first equation is $a$ and the derivative of the second equation is $2x$. We need these derivatives to match at $x = 1$, so we need $a = 2$, so $b = 2$.

29) Evaluate the integrals: $\int (9t-4)^{11} dt$, $\int_1^2 \dfrac{4t}{t^2+1} dt$.

For the first integral, we let $u = 9t - 4$ so $du = 9dt$ and $dt = (1/9)du$

$$\int (9t-4)^{11} dt = \frac{1}{9}\int u^{11} du = \frac{u^{12}}{108} + c = \frac{(9t-4)^{12}}{108} + c$$

For the second integral, we let $u = t^2 + 1$ so $du = 2tdt$ and $4tdt = 2du$. Also, when $t = 1$, $u = 2$ and when $t = 2$, $u = 5$, so

$$\int_1^2 \frac{4t}{t^2+1} dt = 2\int_2^5 \frac{du}{u} = 2\ln u|_2^5 = 2(\ln 5 - \ln 2) = 2\ln(5/2)$$

30) Calculate

$$\lim_{x\to 0} \frac{\int_0^x e^{t^2}}{x}$$

This limit is of the form $0/0$ so we can apply L'Hopital. We note that by the fundamental theorem of calculus,

$$\frac{d}{dx}\int_0^x e^{t^2} = e^{x^2}$$

$$\lim_{x\to 0} \frac{\int_0^x e^{t^2}}{x} = \lim_{x\to 0} \frac{e^{x^2}}{1} = e^0 = 1$$

31) Assume that $f(t)$ is a function such that $\int_1^4 f(t)dt = -2$, $\int_0^5 f(t)dt = 3$ and $\int_4^5 f(t)dt = -1$. Find $\int_0^1 f(t)dt$.

$$\begin{aligned} \int_0^1 f(t)dt &= \int_0^5 f(t)dt - \int_1^5 f(t)dt = \int_0^5 f(t)dt - \left[\int_1^4 f(t)dt + \int_4^5 f(t)dt\right] \\ &= 3 - (-2-1) = 6 \end{aligned}$$