Problems for 152H  

1.* Suppose \( F_A(x) = A^2(1 - x)x^A \). Here \( A \) is a positive real number.

a) Sketch \( F_1(x) \), \( F_{10}(x) \), and \( F_{100}(x) \) for \( x \) in \([0, 1]\).

b) Suppose \( g(A) = \int_0^1 F_A(x) \, dx \). Compute \( g(A) \). What is \( \lim_{A \to \infty} g(A) \)?

c) Consider \( x \) in \([0, 1]\) only and let \( H(x) = \lim_{A \to \infty} F_A(x) \). Compute \( H(x) \) and \( \int_0^1 H(x) \, dx \).

d) Parts b) and c) compare \( \lim_{A \to \infty} \int_0^1 F_A(x) \, dx \) and \( \int_0^1 \lim_{A \to \infty} F_A(x) \, dx \). Do the pictures in part a) help you understand the results?

"In mathematics you don’t understand things, you just get used to them." – John von Neumann

2. The parametric curve \[
\begin{align*}
x &= \sqrt{3} \sin t + \frac{1}{2} \cos t \\
y &= -\sin t + \frac{\sqrt{3}}{2} \cos t
\end{align*}
\]
is a tilted ellipse whose graph is shown to the right. What are the dimensions and location of the box containing the ellipse?

**Note** The sides of the box are vertical and horizontal and also are tangent to the ellipse. The box is called a “bounding box”.

3. If \( A \) is a positive number, the parametric curve \[
\begin{align*}
x &= t^3 - t \\
y &= \frac{A}{1 + t^2}
\end{align*}
\]
looks like:

The “self-intersection” of the curve is perpendicular for one value of \( A \). The picture is this:

Find that value of \( A \).

* Integrals and limits don’t have to work well together. Indeed, they frequently do not. Finding a simple example to compute and show this is not totally easy. What’s here is something “easy”, maybe.
4. The thread length for a simple spool of cotton thread is 25 yards. To celebrate Valentine’s Day, purchase a spool of red thread and send it to your beloved with these instructions:

Unwind the thread and arrange it in the shape of a cardioid, \( r = A(1 - \sin \theta) \).
The area of that cardioid represents how much I love you compared to the ordinary Valentine’s Day card!

Compute the arclength of \( r = A(1 - \sin \theta) \) and find \( A \) so that the length is 25 yards. Then compute the area inside that cardioid. Sketch the result.

**Comment** An opened standard greeting card seems to have area about 70 square inches, or about .054 square yards. On sale, a spool of thread costs about 25 cents. A card these days costs several dollars. Isn’t the thread more cost-effective?

5. A window is in the shape of a 2’ by 3’ rectangle surmounted by a semicircle. Describe the boundary of the window as \( r = f(\theta) \) in a polar coordinate system whose location and orientation you specify. The \( f(\theta) \) likely will need to be defined “piecewise”.

**Comment** A calculus course traditionally has a problem concerning this sort of window, including the rather uncommon word “surmount” as part of the description. Tradition is honored here!