A few more old exam problems

1. Compute $\int_0^\infty (5x+7)\, e^{-x} dx$.

Note The answer is an integer. In the computation you will need to apply L'Hôpital's rule. Please be sure to indicate where you are doing this and why L'Hôpital's rule applies.

Source A spring 2007 Math 152 exam in a course I taught. An answer is available **OVER**.

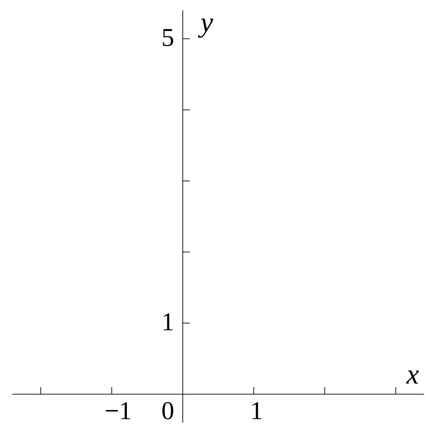
2. Compute $\int_1^\infty \frac{\ln x}{x^3}\, dx$.

Note The answer is a rational number (a quotient of integers). In the computation you will need to apply L'Hopital's rule. Please be sure to indicate where you are doing this and why L'Hopital's rule applies.

Source A spring 2007 Math 152 exam in a course I taught. An answer is available **OVER**.

3. *Part* of a problem from a fall 2005 exam in a course I taught (no answer is available – it was a final exam):

To set the stage (?), previous parts of this problem deal with the curves $y = \frac{2}{3}x + 3$ and $y = \frac{1}{3}x^2 + \frac{1}{3}$. Students were asked to sketch these curves and then find all points of intersection. So you should do this *first* and then have the answer in mind when you try c) below.



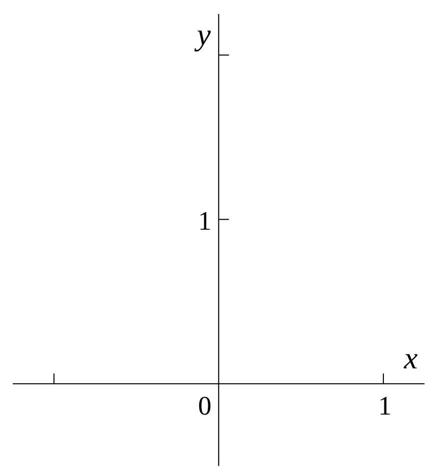
c) The motion of two points, $\mathcal{A}$ and $\mathcal{B}$, is described by these parametric equations:

$$\mathcal{A}: \begin{cases} x = \boldsymbol{t^3} \\ y = \frac{2}{3}\boldsymbol{t^3} + 3 \end{cases} \qquad \mathcal{B}: \begin{cases} x = \mathbf{cos}\, \boldsymbol{t} \\ y = \frac{1}{3}\left(\mathbf{cos}\, \boldsymbol{t}\right)^2 + \frac{1}{3} \end{cases}$$

Sketch the paths of these points as well as possible on the axes to the right. Label the paths with $\mathcal{A}$ and $\mathcal{B}$.

d) Do the points ever collide? Explain your answer briefly.

4. A problem from a fall 2005 exam in a course I taught (no answer is available – it was a final exam):



a) Sketch these polar curves on the axes given:
the cardioid $r = 1 + \sin\theta$ and the circle $r = \sin\theta$.

b) Find the area which is inside the cardioid and outside the circle.

Comment Be *very* careful of the integration(s). Take advantage of symmetry, but the pictures, the areas, and the limits may not be related easily.

OVER

Answers (when available)

1. Compute $\int_0^\infty (5x+7)\,e^{-x}dx$.

Answer We integrate by parts to get $\int (5x+7)\,e^{-x}dx$. Take $u=5x+7$ and $dv=e^{-x}dx$, then $du=5dx$ and $v=-e^{-x}$. So $\int (5x+7)\,e^{-x}dx=(5x+7)(-e^{-x})+\int e^{-x}5dx=-(5x+7)e^{-x}-5e^{-x}+C=-(5x+12)e^{-x}+C$. Now $\int_0^\infty (5x+7)\,e^{-x}dx=\lim\limits_{A\to\infty}\int_0^A (5x+7)\,e^{-x}dx$. And $\int_0^A (5x+7)\,e^{-x}dx=-(5x+12)e^{-x}\Big|_0^A=-(5A+12)e^{-A}+12e^0$. Notice that $\lim\limits_{A\to\infty}(5A+12)e^{-A}=\lim\limits_{A\to\infty}\frac{5A+12}{e^A}\overset{\text{L'H}}{=}\lim\limits_{A\to\infty}\frac{5}{e^A}=0$ where L'Hôpital's rule is used since both $5A+12$ and $e^A\to\infty$ as $A\to\infty$. The value of the integral is therefore $12e^0=12$.

2. Compute $\int_1^\infty \frac{\ln x}{x^3}\,dx$.

Answer Use integration by parts to get an antiderivative of $\frac{\ln x}{x^3}$. Here $u=\ln x$ and $dv=\frac{1}{x^3}\,dx$ so $du=\frac{1}{x}\,dx$ and $v=-\frac{1}{2x^2}$. Then $uv-v\,du$ is $-\frac{\ln x}{2x^2}-\int -\frac{1}{2x^3}dx=-\frac{\ln x}{2x^2}-\frac{1}{4x^2}+C$ (three $-$'s are "built into" the last $-$!). Then (for A positive) $\int_1^A \frac{\ln x}{x^3}\,dx=-\frac{\ln x}{2x^2}-\frac{1}{4x^2}\Big]_1^A=-\frac{\ln A}{2A^2}-\frac{1}{4A^2}-\left(-\frac{\ln 1}{2\cdot 1^2}-\frac{1}{4\cdot 1^2}\right)$. Now as $A\to\infty$, certainly $\frac{1}{4A^2}\to 0$. The limit of $\frac{\ln A}{2A^2}$ needs L'H since both $\ln A$ and A^2 go to ∞. But $\lim\limits_{A\to\infty}\frac{\ln A}{2A^2}\overset{\text{L'H}}{=}\lim\limits_{A\to\infty}\frac{\frac{1}{A}}{4A}=\lim\limits_{A\to\infty}\frac{1}{4A^2}=0$. So the limit of $\int_1^A \frac{\ln x}{x^3}\,dx$ as $A\to\infty$ is $-\left(-\frac{\ln 1}{2\cdot 1^2}-\frac{1}{4\cdot 1^2}\right)$ which is $\frac{1}{4}$.