Here are answers that would earn full credit. Other methods may also be valid.

(10) 1. Show that the surface area obtained when the curve $y = \sqrt{x}$ from $x = 0$ to $x = 2$ is revolved around the x-axis is $\frac{13}{3}\pi$. **Answer** We want $\int_0^2 2\pi y\sqrt{1+(y')^2}dx$. Since $y = \sqrt{x}$, we know $y' = \frac{1}{2\sqrt{x}}$ and $1+(y')^2 = 1+\frac{1}{4x}$ so that $y\sqrt{1+(y')^2} = \sqrt{x}\sqrt{1+\frac{1}{4x}} = \sqrt{x+\frac{1}{4}}$. Now we compute: $\int_0^2 2\pi(x+\frac{1}{4})^{1/2}dx = 2\pi(\frac{2}{3})(x+\frac{1}{4})^{3/2}\Big|_0^2 = \frac{4\pi}{3}(2+\frac{1}{4})^{3/2} - \frac{4\pi}{3}(\frac{1}{4})^{3/2} = \frac{4\pi}{3}(\frac{9}{4})^{3/2} - \frac{4\pi}{3}(\frac{1}{8}) = \frac{27\pi}{6} - \frac{\pi}{6} = \frac{13}{3}\pi$.

(10) 2. Assume $m > 0$. Verify that the improper integral $\int_0^\infty xe^{-mx}dx$ converges and that its value is $\frac{1}{m^2}$.
Answer First we get $\int xe^{-mx}dx$ using Integration by Parts. If $u = x$ then $dv = e^{-mx}dx$ so $du = dx$ and $v = -\frac{1}{m}e^{-mx}$ so $uv - \int v\,du = -x\left(\frac{e^{-mx}}{m}\right) + \frac{1}{m}\int e^{-mx}dx = -x\left(\frac{e^{-mx}}{m}\right) - \frac{e^{-mx}}{m^2} = -\frac{mx+1}{m^2e^{mx}}$. Therefore $\int_0^B xe^{-mx}dx = -\frac{mx+1}{m^2e^{mx}}\Big|_0^B = -\frac{mB+1}{m^2e^{mB}} + \frac{1}{m^2}$. Now consider $\lim_{B\to\infty}\frac{mB+1}{m^2e^{mB}}$. Since $m > 0$, $\lim_{B\to\infty} mB+1 = \infty$ and $\lim_{B\to\infty} e^{mB} = \infty$. The limit is eligible for L'H. We $\frac{d}{dB}$ the top and bottom, and get $\lim_{B\to\infty}\frac{m}{m^3e^{mB}}$. This is 0 since we have a constant on top and exponential growth on the bottom. The improper integral must converge, and its value is $\frac{1}{m^2}$. (This is called the **Laplace transform** of x.)

(10) 3. a) Does the sequence defined by the formula $a_n = (7n^3+5)^{(2/n)}$ converge? If it does, find its limit. If it does not, explain why. **Answer** Consider $\ln a_n = \left(\frac{2}{n}\right)\ln(7n^3+5) = \frac{2\ln(7n^3+5)}{n}$. What happens as $n \to \infty$? The top and bottom both $\to \infty$, so we use L'H. The bottom becomes 1, so we get $\lim_{n\to\infty}\frac{2(21n^2)}{7n^3+5}$. Now either by degree considerations (top has degree 2, which is < 3, the degree of the bottom) or by repeated use of L'H, we see that the limit is 0. Exponentiate to get the limit of the original sequence: $e^0 = 1$.

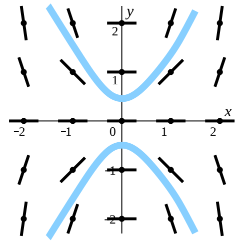
b) Does the sequence defined by the conditions $\begin{cases} b_1 = 1 \\ b_{n+1} = b_n + \frac{1}{b_n} & \text{if } n \ge 1 \end{cases}$ converge? If it does, find its limit. If it does not, explain why. **Answer** If the sequence converges, call the limit L. Then $L \ge 1$ since all of the terms of the sequence are ≥ 1 (the first term *is* 1, and the sequence is increasing). We know that $\lim_{n\to\infty} b_{n+1}$ is also L and $\lim_{n\to\infty}\frac{1}{b_n} = \frac{1}{L}$ (we need $L \ne 0$ for the second limit). Therefore $L = L + \frac{1}{L}$ but there is no number L so that $\frac{1}{L} = 0$. The limit of the sequence does *not* exist.

(8) 4. Bruno and Igor are again sharing a loaf of bread. Bruno, now hungrier and more ferocious, eats two-thirds of the loaf, then Igor eats eats half of what remains, then Bruno eats two-thirds of what remains, then Igor eats half of what remains, and so on. How much of the loaf will each student eat?
Answer I'll compute the first two "rounds". Bruno eats $\frac{2}{3}$, and passes Igor $\frac{1}{3}$. Igor eats $\frac{1}{6}$ and passes $\frac{1}{6}$. Bruno eats $(\frac{2}{3})\frac{1}{6} = \frac{1}{9}$ and passes $(\frac{1}{3})\frac{1}{6} = \frac{1}{18}$. Igor eats $\frac{1}{36}$ and passes $\frac{1}{36}$. So Bruno eats $(\frac{2}{3})\frac{1}{36} = \frac{1}{18\cdot 3} = \frac{1}{54}$. Let's consider Bruno's eating: $\frac{2}{3}$, $\frac{1}{9}$, and $\frac{1}{54}$. These are the initial terms of a geometric series with first term $\frac{2}{3}$ and ratio $\frac{1}{6}$. So Bruno eats $\frac{\frac{2}{3}}{1-\frac{1}{6}} = \frac{\frac{2}{3}}{\frac{5}{6}} = \frac{12}{15} = \frac{4}{5}$ of the bread and Igor eats $1 - \frac{4}{5} = \frac{1}{5}$. (You can check independently that Igor also eats a geometric series with sum $\frac{1}{5}$.)

(10) 5. I know that $\sum_{n=1}^{\infty}\frac{1}{5^n+3^n} \approx .162$ with error ($\pm$) less than .001. Find N so that the partial sum $\sum_{n=1}^{N}\frac{1}{5^n+3^n}$ is guaranteed to be within $.001 = \frac{1}{1{,}000}$ of the sum of the infinite series. **Answer** Since $\frac{1}{5^n+3^n} < \frac{1}{5^n}$ we want N so that $\sum_{n=N+1}^{\infty}\frac{1}{5^n} < \frac{1}{1{,}000}$. Here $\sum_{n=N+1}^{\infty}\frac{1}{5^n}$ is a geometric series with first term $\frac{1}{5^{N+1}}$ and ratio $\frac{1}{5}$. Its sum is $\frac{\frac{1}{5^{N+1}}}{1-\frac{1}{5}} = \frac{1}{4\cdot 5^N}$. Take $N = 4$ since $5^4 = 625$.

(8) 6. Solve the initial value problem: $y^2\frac{dy}{dx} = x^{-3}$ and $y(2) = 0$. In the answer express y explicitly as a function of x.
Answer This is a separable differential equation. We get $\int y^2\,dy = \int x^{-3}dx$ so that $\frac{1}{3}y^3 = -\frac{1}{2}x^{-2} + C$. We use the initial condition and get $0 = -\frac{1}{8} + C$ so $C = \frac{1}{8}$. The solution becomes $\frac{1}{3}y^3 = -\frac{1}{2}x^{-2} + \frac{1}{8}$. We solve to get $y = \left(-\frac{3}{2}x^{-2} + \frac{3}{8}\right)^{1/3}$.

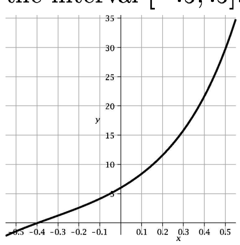
(10) 7. a) Sketch the slope field of $\frac{dy}{dx} = xy$ for $-2 \le x \le 2$ and $-2 \le y \le 2$ at the indicated points. **Answer** That's done to the right, along with two solution curves. The solution curves, not requested, may help with the answers to the other parts of the question.



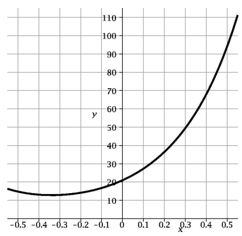
b) Based on the sketch, determine $\lim_{x\to\infty} y(x)$, where $y(x)$ is a solution with $y(0) > 0$.
Answer The limit is $\underline{+\infty}$

c) Based on the sketch, determine $\lim_{x\to\infty} y(x)$, where $y(x)$ is a solution with $y(0) < 0$.
Answer The limit is $\underline{-\infty}$

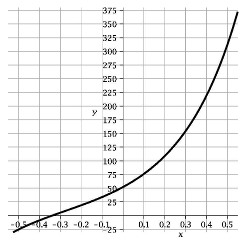
(12) 8. Suppose $f(x) = e^{x^2+\sin x}$. Here are values of f and some of its derivatives at 0: $f(0) = 1$; $f'(0) = 1$; $f''(0) = 3$; $f^{(3)}(0) = 6$; $f^{(4)}(0) = 21$; $f^{(5)}(0) = 52$. Below are graphs of $f^{(3)}(x)$, $f^{(4)}(x)$, and $f^{(5)}(x)$ on the interval $[-.5, .5]$.



Graph of $f^{(3)}(x)$



Graph of $f^{(4)}(x)$



Graph of $f^{(5)}(x)$

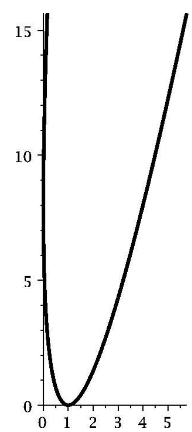
Assume this information is correct. No additional computation of the values of f or any of its derivatives is needed for this problem.

a) What is the second degree Taylor polynomial centered at 0 of f? *Do no unnecessary arithmetic!* **Answer** $T_2(x) = 1 + 1x + \frac{3}{2}x^2$

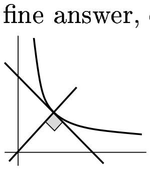
b) Find a polynomial $P(x)$ so that $|P(x) - f(x)| < .01$ for all x in the interval $\left[-\frac{1}{4}, \frac{1}{4}\right]$. You should write the polynomial and explain why the error is less than $.01 = \frac{1}{100}$.

Answer Candidates are Taylor polynomial for f centered at 0 of degree $n = 2$, 3, or 4. We need to estimate $K\frac{|x-a|^{n+1}}{(n+1)!}$. Here $a = 0$ and x is in $\left[-\frac{1}{4}, \frac{1}{4}\right]$ so that $|x-a|^{n+1} \le \left(\frac{1}{4}\right)^{n+1} = \frac{1}{4^{n+1}}$. K is an overestimate of $|f^{(n)}(x)|$ for x in $\left[-\frac{1}{4}, \frac{1}{4}\right]$, and this can be gotten by inspecting the graphs given. Of course $\frac{1}{4} = .25$. $\underline{n=2}$: Here $K \le 15$ so the error is estimated by $\frac{15}{3!4^3} = \frac{15}{6\cdot 64}$ which is *not* $\le \frac{1}{100}$. $\underline{n=3}$: Here $K \le 50$ so the error is estimated by $\frac{50}{4!4^4} = \frac{50}{24\cdot 256}$ which *is* $\le \frac{1}{100}$. (Just consider $\frac{50}{24} \overset{?}{\le} \frac{256}{100}$ so that's $5{,}000 \le 24\cdot 256$ and *TRUE*.) So take $P(x)$ to be $T_3(x) = 1 + 1x + \frac{3}{2}x^2 + \frac{6}{6}x^3$. $\underline{n=4}$: K is ≤ 150 and the relevant fraction is $\frac{150}{5!4^5} = \frac{150}{120\cdot 1{,}024}$ which is also *is* $\le \frac{1}{100}$ so $P(x) = T_3(x) = 1 + 1x + \frac{3}{2}x^2 + \frac{6}{6}x^3 + \frac{21}{24}x^4$ is also valid.

(10) 9. Part of the graph of the parametric curve $\begin{cases} x = t^2 \\ y = 8(1-t)^2 \end{cases}$ is shown to the right. Find the (x, y) coordinates of the point on the curve which is closest to the origin.

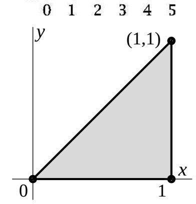


Answer Minimize the distance from (x, y) to $(0, 0)$. Consider $\sqrt{x^2+y^2} = \sqrt{t^4 + 8^2(1-t)^4}$. Then $\frac{d}{dt}$ this and set the result equal to 0. The result is $4t^3 - 8^2\cdot 4(1-t)^3 = 0$ so $t^3 = 8^2(1-t)^3$. Cube root gives us $t = 2^2(1-t) = 4 - 4t$ so that $t = \frac{4}{5}$ and the point (x, y) is $\left(\left(\frac{4}{5}\right)^2, 8^2\left(\frac{1}{5}\right)^2\right)$, fine answer, or, if you insist, $\left(\frac{16}{25}, \frac{64}{25}\right)$.



Another method is to locate the closest point by considering the slope of the tangent line. The tangent line should be perpendicular to the line segment joining (x, y) and $(0, 0)$. See the picture to the left. That line segment has slope $\frac{y}{x}$, and the tangent line has slope $\frac{dy/dt}{dx/dt}$. At the minimum we should have $\frac{y}{x} = -\frac{dx/dt}{dy/dt}$ and this is $\frac{8(1-t)^2}{t^2} = -\frac{2t}{-8\cdot 2(1-t)}$ which leads to the equation we had before.

(12) 10. Suppose that T is the triangular region shown with vertices at $(0, 0)$, $(0, 1)$, and $(1, 1)$.



a) Describe T using polar coordinates. **Answer** The boundary is parts of $y = 0$, $y = x$, and $x = 1$. This is $\theta = 0$, $\theta = \frac{\pi}{4}$, and $r\cos\theta = 1$. Points in T have $0 \le \theta \le \frac{\pi}{4}$ and $0 \le r \le \frac{1}{\cos\theta}$.

b) Compute the area of T using polar coordinates. **Answer** $\frac{1}{2}\int_0^{\frac{\pi}{4}} r^2 d\theta = \frac{1}{2}\int_0^{\pi/4}\left(\frac{1}{\cos\theta}\right)^2 d\theta$ $= \frac{1}{2}\int_0^{\pi/4}(\sec\theta)^2 d\theta = \frac{1}{2}\tan\theta\Big|_0^{\pi/4} = \frac{1}{2}(1-0) = \frac{1}{2}$ (as it *should* be).

Comment for problem 8 To the right are graphs of $f - T_2$, $f - T_3$, and $f - T_4$, respectively, on $\left[-\frac{1}{4}, \frac{1}{4}\right]$ with dashed lines at the $\pm.01$ heights. T_2 certainly fails the approximation requirement, but T_3 and T_4 seem good enough.