1. Suppose that \( f(x) = x^{3/2} \).
   a) Find the third degree Taylor polynomial, \( T_3(x) \), centered at \( c = 4 \) for \( f(x) \).

   **Answer** If \( f(x) = x^{3/2} \) then \( f'(x) = \frac{3}{2}x^{1/2} \), \( f''(x) = \frac{3}{4}x^{-1/2} \), and \( f^{(3)}(x) = -\frac{3}{8}x^{-3/2} \), so \( f(4) = 8 \), \( f'(4) = 3 \), \( f''(4) = \frac{3}{8} \), and \( f^{(3)}(4) = -\frac{3}{32} \). Then \( T_3(x) = 8 + 3(x-4) + \frac{3}{8}(x-4)^2 - \frac{3}{32}(x-4)^3 \). (Don’t forget factorials!)
   b) Suppose \( T_3(x) \) is the polynomial found in a). Use Taylor’s inequality (the **Error Bound**) to find an overestimate for \( |f(x) - T_3(x)| \) on the interval \([2, 6]\). Your answer should be an explicit number valid for every \( x \) on this interval.

   **Answer** Here \( n = 3 \) and \( c = 4 \). If \( x \) is in \([2, 6]\), \(|x-c|\) is at most 2. Also \( |f^{(4)}(x)| = \left( \frac{9}{160} \right)x^{-5/2} \). This function is **decreasing** either because it is (positive)\(x^{\text{negative}}\) or because its derivative is negative so the maximum of \( f^{(4)}(x) \) on \([2, 6]\) is \( f^{(4)}(2) = \left( \frac{9}{160} \right)2^{-5/2}/24 \) (don’t forget the factorial: \( 4! = 24 \)). This can be “simplified” (I wouldn’t!) to \( \frac{3}{64\sqrt{2}} \).

   **Comment** This overestimate is \( \approx 0.06629 \). The graph of \(|f(x) - T_3(x)|\) on \([2, 6]\) displayed to the right supports the overestimate but shows it is not “sharp”.

2. a) Suppose that \( f(x) = Ce^{(x^2)} + x \) (here \( C \) is an undetermined constant). Verify that \( f(x) \) is a solution of the differential equation \( y' = 2xy - 2x^2 + 1 \).

   **Answer** Since \( f'(x) = Ce^{(x^2)} \cdot 2x + 1 \) and \( 2xf(x) - 2x^2 + 1 = 2x(Ce^{(x^2)} + x) - 2x^2 + 1 = 2xCe^{(x^2)} + 2x^2 - 2x^2 + 1 = 2xCe^{(x^2)} + 1 \) is the same, \( y = f(x) \) is a solution.
   b) Find a solution of the differential equation \( y' = 2xy - 2x^2 + 1 \) which passes through the point \((2, 3)\).

   **Answer** Since \( f(2) = 3 \) we know \( f(2) = Ce^4 + 2 = 3 \) and \( C = \frac{1}{e^4} \). The desired solution is \( y = \left( \frac{1}{e^4} \right)e^{(x^2)} + x \).

3. This problem is about the differential equation \( y' = y(1 - \frac{1}{4}y) \).
   a) Find the equilibrium solutions (where \( y \) doesn’t change) for this differential equation. **Answer** \( y = 0 \) and \( y = 2 \) and \( y = -2 \).
   b) To the right is a direction field for this equation.

   Sketch solution curves on this graph through the points below and find the indicated limits:
   
   - **(0, 1)**. **Label this curve A.** On curve **A**, \( \lim_{x \to -\infty} y(x) = 0 \) and \( \lim_{x \to +\infty} y(x) = \frac{2}{e^2} \).
   - **(0, −1)**. **Label this curve B.** On curve **B**, \( \lim_{x \to -\infty} y(x) = 0 \) and \( \lim_{x \to +\infty} y(x) = -2 \).

   **Comment** These curves were drawn with numerical approximation methods using **Maple**. Students do similar work in Math 244.

4. The infinite series \( \sum_{n=1}^{\infty} \frac{3}{2\sqrt{n+4}} \) converges and its sum, to an accuracy of .001, is .719. Find a positive integer \( N \) so that the partial sum, \( S_N = \sum_{n=1}^{N} \frac{3}{2\sqrt{n+4}} \), has a value within .001 of the sum of the whole series. Explain your reasoning.

   **Answer** Certainly \( \sum_{n=1}^{\infty} \frac{3}{2\sqrt{n+4}} = S_N + T_N \) and we overestimate \( T_N \). Since \( 0 < \frac{3}{2\sqrt{n+4}} < \frac{3}{2\sqrt{N+4}} \), an overestimate is \( \sum_{n=N+1}^{\infty} \frac{3}{2\sqrt{n+4}} = \frac{3}{2\sqrt{N+4}} \) (geometric series) which is \( \frac{1}{N} \). A suitable value of \( N \) is 5 (from the table given).

5. The infinite series \( \sum_{n=1}^{\infty} \frac{1}{5n^3 + n^2} \) converges and its sum, to an accuracy of .001, is .205. Find \( N \) so that the partial sum, \( S_N = \sum_{n=1}^{N} \frac{1}{5n^3 + n^2} \), has a value within .001 of the sum of the whole series. Explain your reasoning.
Answer Again \(\sum_{n=1}^{\infty} \frac{1}{n^{1/2} + n} = S_N + T_N\). \(T_N\) can be overestimated by \(\sum_{n=N+1}^{\infty} \frac{1}{3n^2}\), which is the area of a collection of rectangles. The \(n^{th}\) rectangle has lower side on the interval \([n, n + 1]\) and sits under, touching at one point, the graph of \(y = f(x)\) where \(f(x) = \frac{1}{\sqrt{x}}\). So \(T_N < \int_{N}^{\infty} \frac{1}{\sqrt{x}} \, dx\). Compute this improper integral: \(\int_{N}^{\infty} \frac{1}{\sqrt{x}} \, dx = \lim_{B \to \infty} \int_{N}^{B} \frac{1}{\sqrt{x}} \, dx = \frac{1}{B^{1/2}} + \frac{1}{N^{1/2}} = \frac{1}{10^{1/2}}\). \(T_N\) is less than .001 if \(N = 10\).

(12) 6. a) Suppose the sequence \(\{a_n\}\) is defined by \(a_n = (7n + 3)^{5/n}\). Find the exact value of the limit of this sequence. Answer If \(a_n = (7n + 3)^{5/n}\), then \(\ln(a_n) = \frac{5}{n} \ln(7n + 3)\). The top and bottom of this fraction both \(\to \infty\) as \(n \to \infty\), so this expression is eligible for L'H: \(\lim_{n \to \infty} \frac{5 \ln(7n + 3)}{n} = \lim_{n \to \infty} \frac{5 - 3}{\sqrt{n}} = 0\). The limit of the original sequence is \(e^0 = 1\).

Comment Convergence is not very fast. The sequence value is 1.04526 when \(n = 1,000\).

b) It is known that the sequence \(\sqrt{3}, \sqrt{3 + \sqrt{3}}, \sqrt{3 + \sqrt{3 + \sqrt{3}}}, \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3}}}}, \ldots\) converges. Find the exact value of the limit of this sequence. Answer If \(b_n\) is the \(n^{th}\) term of this sequence, then \(b_{n+1} = \sqrt{3 + b_n}\). Therefore \((b_{n+1})^2 = 3 + b_n\). Now take the limit as \(n \to \infty\), and suppose \(L\) is the limit of the sequence. Then \(L^2 = 3 + L\) or \(L^2 - L - 3 = 0\) so (quadratic formula) \(L = \frac{1 + \sqrt{13}}{2}\). Since all of the terms of the sequence are positive, the limit will be non-negative, so it is \(\frac{1 + \sqrt{13}}{2}\).

Comment The limit is \(\approx 2.30277\). The \(10^{th}\) term agrees with this to more than 5 decimal places.

7. There is an infinite sequence of circles which do not overlap and which have radius 1, \(\frac{1}{3}\), \(\frac{1}{3^2}\), \ldots as shown. (The \(n^{th}\) circle has radius \(\frac{1}{n}\)) a) Is the total area inside all of the circles finite? Answer The \(n^{th}\) circle has area \(\pi \left(\frac{1}{n}\right)^2\). The total area is \(\pi \sum_{n=1}^{\infty} \frac{1}{n^2}\), a p-series with \(p = 2 > 1\) which converges: the total area is finite.

b) Is the total circumference of all of the circles finite? Answer The circumference of the \(n^{th}\) circle is \(2\pi \left(\frac{1}{n}\right)\) so the total area is \(2\pi \sum_{n=1}^{\infty} \frac{1}{n}\). This is a p-series with \(p = 1\) (actually, the harmonic series), so it diverges: the total circumference is infinite.

8. Does the infinite series \(\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)^2}{(2n)!}\) converge or diverge? Answer First try absolute convergence. Consider a series whose \(n^{th}\) term \(c_n\) is \(\frac{(n+1)^2}{(2n)!}\) and use the Ratio Test: \(\frac{c_{n+1}}{c_n} = \frac{(n+2)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n+1)^2}\). Since \((n + 1)! = (n + 1)n!\) and \((2(n + 1))! = (2n + 2)! = (2n + 2)(2n + 1)(2n)!\), the fraction becomes \(\frac{(n+2)^2}{(2n+2)(2n+1)}\) by dividing the top and bottom by \(n^2\). Then \(\lim_{n \to \infty} \frac{(n+2)^2}{(2n+2)(2n+1)} < 1\). The Ratio Test shows the original series converges absolutely, and therefore it must converge. (L'H could also be used.)

9. Suppose \(a\) and \(b\) are unknown positive numbers. When the line segment connecting the origin to the point \((a, b)\) is revolved about the \(x\)-axis, the result is the slanted surface of a cone. The area of this surface is \(\pi b\sqrt{a^2 + b^2}\). Verify this formula using calculus. Answer The line segment is the graph of \(y = \left(\frac{b}{a}\right)x\) over \([0, a]\). Therefore the surface area is \(2\pi \int_{0}^{a}\sqrt{1 + \left(\frac{b}{a}\right)^2} \, dx = 2\pi \int_{0}^{a} \sqrt{1 + \left(\frac{b}{a}\right)^2} \, x \, dx = 2\pi \left(\frac{b}{a}\right) \sqrt{1 + \left(\frac{b}{a}\right)^2} \left(\frac{a^2}{2}\right) = \pi b\sqrt{a^2 + b^2}\).

Brief answers to version B (the questions which are different)

1. \(T_3(x) = 243 + \frac{15}{16} (x - 9) + \frac{45}{64} (x - 9)^2 + \frac{5}{384} (x - 9)^3\); the overestimate is \(\left(\frac{15}{16}\right)^{3/2} \left(\frac{45}{64}\right)^{3/2} (x - 9)^3\) (this is \(\frac{5067}{5067}\) but I hope no one writes this!).

2. a) is much the same. In b), \(C = \frac{1}{2}\).

4. \(N \geq 7\). The reasoning is similar.

5. \(N \geq 10\) here also. The reasoning is similar.

6. a) The limit is 1. b) The limit is \(\frac{1 + \sqrt{13}}{2}\). The reasoning in both parts is similar.