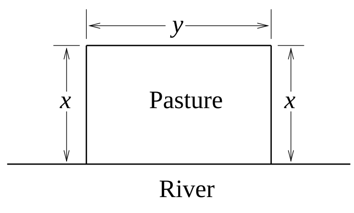
15. Area A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 180,000 square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river?

Answer We sketch the river and the pasture below.

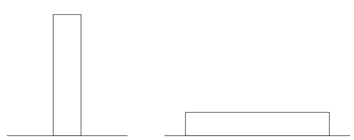


Suppose y is the length in meters of the side of the rectangle parallel to the river, and x is the length in meters of the side of the rectangle perpendicular to the river. The total length, L, of fencing needed is given by $L = 2x + y$, while the area enclosed, A, is given by $A = xy$.

Since the area is given, we have the constraint $180,000 = xy$. Physically, both x and y must be positive, so we can solve this constraint to obtain $y = \frac{180,000}{x}$. Then L, the objective function which we must minimize, can be written as a function of x alone:

$$L(x) = 2x + \frac{180,000}{x}$$

and we are left with the job of *minimizing* $L(x)$ when $x \in (0, +\infty)$. Note that when x is very large positive, the first term in $L(x)$ is large. That is, even though the rectangle has fixed area, it may have very long sides perpendicular to the river. When x is small and positive, y, the second term in the formula for $L(x)$, becomes large and positive. Approximate sketches of such "extreme" rectangles are shown below (not using the same scale as the picture above!).



x very large x very small

We know that $\lim_{x \to 0^+} L(x) = +\infty$ and $\lim_{x \to +\infty} L(x) = +\infty$. The minimum of the differentiable function L must occur therefore at some point inside the interval $(0, +\infty)$. This minimum will occur at a critical point. Computation yields

$$L'(x) = 2 - \frac{180,000}{x^2}$$

and $L'(x) = 0$ when $180,000 = 2x^2$, or when $x^2 = 90,000$ so that x, which is positive, must be 300. Since this is the only critical point of L and we know that L gets large near the "edges" of its domain, it must be true that L achieves its minimum at $x = 300$. Computation shows that $L(300) = 1,200$ and we conclude that the minimum amount of total fencing needed is 1,200 feet. The dimensions which give that minimum are $x = 300$ feet (perpendicular to the river) and $y = 600$ feet (parallel to the river).