SB: Foundations of Calculus **HW #5, due 7/9/2013**

The Intermediate Value Theorem Suppose f is a continuous function on the closed interval $[a, b]$. Then the equation $f(x) = K$ has at least one solution in $[a, b]$ whenever K is between $f(a)$ and $f(b)$.

The Bisection Algorithm (a way to approximate a root of $f(x) = 0$)

Entry conditions
A continuous function $f(x)$ defined on an interval $[a, b]$, with $f(a) \cdot f(b) \leq 0$; a positive tolerance E for the error.

Output
An interval $[c, d]$ so that $d - c < E$ and $f(c) \cdot f(d) \leq 0$. This identifies an interval of length less than E which must contain a root of $f(x) = 0$.

"Loop" structure
Given $[p, q]$ with $f(p) \cdot f(q) \leq 0$, let $m = (p + q)/2$. Compute $f(m)$. If $f(m) \cdot f(p) \leq 0$, then replace q by m; else replace p by m.

Exit check If $q - p < E$ then return p and q as c and d in the output description; else go to loop.

Please hand in solutions to problems Q1, Q2, and Q3 below.

Q1. The first twenty-five decimal digits (accurately rounded, both sides of the decimal point!) of the cube root of 2 are 1.25992 10498 94873 16476 7211. Describe how the Bisection Algorithm can be used to produce just the first three digits (the 1.25). *You* specify the continuous function f to be used, the initial interval $[a, b]$, and the positive output tolerance E. Then state, at each stage, the progress of the algorithm: the m and how it replaces one of p and q (which one in each step). Use of a calculator is strongly recommended.

Q2. a) Suppose that $f(x) = x^2$ and $g(x) = 2^x$. Compute $f(-2)$, $g(-2)$, $f(5)$, and $g(5)$. According to the Intermediate Value Theorem (assuming both f and g are continuous, which is in fact true) and the function values computed, what is the smallest number of roots the equation $f(x) = g(x)$ can have?

b) Suppose still that $f(x) = x^2$ and $g(x) = 2^x$. Graph $y = f(x)$ and $y = g(x)$ carefully on the interval $-2 \leq x \leq 5$. How many roots does the equation $f(x) = g(x)$ appear to have?

c) Draw graphs of two increasing continuous functions which intersect exactly two times.

d) Draw graphs of two increasing continuous functions which intersect exactly three times.

e) Draw graphs of two increasing continuous functions which intersect exactly four times.

Q3. Suppose $f(x) = x^3$ and $g(x) = 4\cos(7x + 5) + 8\sin(x^2 - 9) + 6$. Does the equation $f(x) = g(x)$ have at least one solution (you may assume known that f and g are continuous)? Give an interval in which a solution of $f(x) = g(x)$ can be found. Explain your reasoning using complete English sentences.

Hint How big and how small can g be? (You don't need to be too precise!)

What's an algorithm?

A precise definition of **algorithm** is difficult, which is interesting since the concept has become central to much of mathematics and computer science during the last quarter century. It is as vital and important to such study as the **sonnet** is to the history and practice of poetry. Here are some quotes from Knuth's book *The Art of Computer Programming*. (Donald E. Knuth is probably the world's most famous computer scientist.)

From page 1:

> The word "algorithm" itself is quite interesting; at first glance it may look at though someone intended to write "logarithm" but jumbled up the first four letters. ... the true origin of the word ... comes from the name of a famous Persian textbook author, Abu Ja'far Mohammed ibn Mûsâ al-Khowârismî (*c.* 825) – literally, "father of Ja'far, Mohammed, son of Moses, native of Khowârizm." Khowârizm is today the small Soviet city of Khiva. Al-Khowârizmî wrote the celebrated book *Kitab al jabr w'al-muqabala* ("Rules of restoration and reduction"); another word, "algebra", stems from the title of his book, although the book wasn't really very algebraic.

From pages 4, 5, and 6:

> The modern meaning for algorithm is quite similar to that of *recipe, process, method, technique, procedure, routine,* except that the word "algorithm" connotes something just a little different. Besides merely being a finite set of rules which gives a sequence of operations for solving a specific type of problem, an algorithm has five important features:
>
> **1) Finiteness.** An algorithm must always terminate after a finite number of steps. ...
>
> **2) Definiteness.** Each step of an algorithm must be precisely defined; the actions to be carried out must be rigorously and unambiguously specified for each case. ...
>
> **3) Input.** An algorithm has zero or more inputs, i.e., quantities which are given to it initially before the algorithm begins. These inputs are taken from specified sets of objects. ...
>
> **4) Output.** An algorithm has one or more outputs, i.e., quantities which have a specified relation to the inputs. ...
>
> **5) Effectiveness.** An algorithm is also generally expected to be *effective*. This means that all of the operations to be performed in the algorithm must be sufficiently basic that they can in principle be done exactly and in a finite length of time ...

Knuth continues on the same page to contrast his definition of algorithm with what could be found in a cookbook:

> Let us try to compare the concept of an algorithm with that of a cookbook recipe: A recipe presumably has the qualities of finiteness (although it is said that a watched pot never boils), input (eggs, flour, etc.) and output (TV dinner, etc.) but notoriously lacks definiteness. There are frequently cases in which the definiteness is missing, e.g., "Add a dash of salt." A "dash" is defined as "less than $\frac{1}{8}$ teaspoon"; salt is perhaps well enough defined; but where should the salt be added (on top, side, etc.)? ...

He concludes his comparison by writing:

> ... a computer programmer can learn much by studying a good recipe book.