B1 Suppose \( K \) is a compact non-empty subset of \( \mathbb{C} \). Show that there is \( T \in L(H) \) so that \( \sigma(T) = K \).

B2 The Volterra operator, \( V: L^2([0, 1]) \to L^2([0, 1]) \), is defined by \( Vf(x) = \int_0^x f(t) \, dt \).

a) What is \( V^* \)? \( V + V^* \)? What is the image of \( V + V^* \)?

Remark This is problem 7 of section 2.2 in Conway’s *A Course in Functional Analysis*.

b) What is \( \sigma_p(V) \), the collection of eigenvalues of \( V \)?

B3 (Continuing the preceding problem.) What can you say about \( \sigma(V) \)? (You will probably need the Spectral Radius Formula and inductive discussion of \( V^n \) and \( \|V^n\| \).)

B4 (Hilbert matrix) Show that \( (Ae_j, e_i) = (i+j+1)^{-1} \) for \( 0 \leq i,j < \infty \) defines a bounded operator on \( \ell^2(\mathbb{N} \cup \{0\}) \) (square-summable sequences beginning with the index 0) with \( \|A\| \leq \pi \).

Remark This is problem 10 of section 2.1 in Conway’s *A Course in Functional Analysis*. The problem statement there contains further references.

B5 If \( H \) is an infinite dimensional Hilbert space, show that no orthonormal basis for \( H \) is a Hamel (vector space) basis. Show that a Hamel basis is uncountable.

B6 Suppose \( H \) is the collection of all absolutely continuous functions \( f(0): [0, 1] \to \mathbb{F} \) with \( f(0) = 0 \) and \( f' \in L^2([0, 1]) \). Let \( \langle f, g \rangle = \int_0^1 f'(t)g'(t) \, dt \).

a) Prove that \( H \) is a Hilbert space.

b) Find an orthonormal basis of \( H \).

Remark This is problem 3 of section 1.1 and problem 4 of section 1.4 in Conway’s *A Course in Functional Analysis*.

B7 Let \( H = \ell^2(\mathbb{N} \cup \{0\}) \) (square-summable sequences beginning with the index 0).

a) Show that if \( \{\alpha_n\} \in H \), then the power series \( \sum_{n=0}^{\infty} \alpha_n z^n \) has radius of convergence \( \geq 1 \).

b) If \( |\lambda| < 1 \) and \( L: H \to \mathbb{F} \) is defined by \( L(\{\alpha_n\}) = \sum_{n=0}^{\infty} \alpha_n \lambda^n \), find the vector \( h_0 \) in \( H \) so that \( L(h) = \langle h, h_0 \rangle \) for all \( h \in H \). What is the norm of \( L \)?

c) Define \( \tilde{L}: H \to \mathbb{F} \) by \( \tilde{L}(\{\alpha_n\}) = \sum_{n=1}^{\infty} n\alpha_n \lambda^{n-1} \), again with \( |\lambda| < 1 \). Now find the corresponding \( \tilde{h}_0 \) so that \( \tilde{L}(h) = \langle h, \tilde{h}_0 \rangle \) for all \( h \in H \).

Remark This is problem 3 and problem 4 of section 1.3 in Conway’s *A Course in Functional Analysis*.

B8 Suppose that \( A \) and \( B \) are self-adjoint. Prove that \( AB \) is self-adjoint if and only \( AB = BA \).

Remark This is problem 11 of section 2.3 in Conway’s *A Course in Functional Analysis*.