Projective space

Most pretentious approach:

Let $V$ be a vector space over a field $F$. Then we put $P(V)$ to be the set of all 1 dimensional subspaces of $V$ (the lines). In particular, $V$ can be chosen to be $L^2(\mathbb{R})$ (important in Quantum Mechanics, among others), or a finite dimensional vector space $F^n$ (important in Combinatorics, for example).

This approach is maybe too difficult to achieve our aim: the study of $CP^1$ - the 1-dimensional complex projective space. So, we will define it first as an equivalence relation, in the following way:

\[
\begin{align*}
C^2 & = \{(\alpha, \beta) : \alpha, \beta \in \mathbb{C}\} \\
C^{2*} & = C^2 - \{(0,0)\} \\
(\alpha_1, \beta_1) \sim (\alpha_2, \beta_2) & \iff \exists \lambda \in \mathbb{C}^* : \lambda(\alpha_2, \beta_2) = (\alpha_1, \beta_1) \\
CP^1 & = C^{2*} / \sim
\end{align*}
\]

The equivalence classes in $CP^1$ are the one dimensional subspaces of $C^2$ over $\mathbb{C}$.

How to define a topology on $CP^1$?

Suppose $(\alpha, \beta) \in C^{2*} \Rightarrow [(\alpha, \beta)] \in CP^1 \Rightarrow [\alpha, \beta] = [\lambda(\alpha, \beta)] \in CP^1$. A unique representative (in some sense) of the class $[\alpha, \beta]$ can be described in the following way:

\[
[\alpha, \beta] = \begin{cases} 
[\beta, 1] & \text{if } \beta \neq 0 \iff [z, 1], z \in \mathbb{C} \\
[1, 0] & \text{if } \beta = 0
\end{cases}
\]

So, we can imagine $CP^1$ as a copy of $\mathbb{C}$ together with a distinct element $[1, 0]$, which intuitively will be $\infty$.

For $z \neq 0$, we also have $[z, 1] = [1, \frac{1}{z}]$. We can use this to define a topology on $CP^1$. For $z \in \mathbb{C}^*$, we make the identification:

\[
z \leftrightarrow [z, 1] \leftrightarrow [1, z] \leftrightarrow \frac{1}{z}
\]

This is a continuous overlap mapping from the open sets $\mathbb{C}^*$ to $\mathbb{C}^*$. Therefore, if we put $C_z = \mathbb{C} \cup \{1, 0\}$ and $C_w = \mathbb{C} \cup \{0, 1\}$ and consider the above mentioned correspondence between $(C_z \cup C_w)/\sim \leftrightarrow CP^1$ we maybe can view $CP^1$ to be homeomorphic to the one point compactification of $\mathbb{C}$. That is $CP^1 \cong \mathbb{C} \cup \{\infty\}$, where the neighborhoods of $\infty$ are of the form $\{z \in \mathbb{C} : |z| > A, A \in \mathbb{R}\}$.

Another way to put a topology on $CP^1$ is to consider the projection:

\[
C^{2*} \overset{\pi}{\longrightarrow} C^{2*} / \sim \cong CP^1
\]
and put a topology on $\mathbb{C}P^1$ such that $\pi$ is continuous. In general, it is taken
the strongest topology in which the mapping $\pi$ remains continuous. This
topology is equivalent to the one just defined.

**Yet another equivalent way** is to consider the stereographic projection
of the Riemann sphere. This would make $\mathbb{C}P^1 \cong S^2$.

We consider the following diagram:

$$(\alpha, \beta) \in \mathbb{C}^2 \to \mathbb{C}P^1 \overset{F}{\to} \mathbb{C}P^1 \to \mathbb{C}^2 \ni (f(\alpha, \beta), g(\alpha, \beta))$$

We would want:

$$(\alpha_1, \beta_1) \sim (\alpha_2, \beta_2) \Rightarrow (f(\alpha_1, \beta_1), g(\alpha_1, \beta_1)) \sim (f(\alpha_2, \beta_2), g(\alpha_2, \beta_2)) \Leftrightarrow$$

$$(\alpha_1, \beta_1) = \lambda(\alpha_2, \beta_2) \Rightarrow (f(\alpha_1, \beta_1), g(\alpha_1, \beta_1)) = \mu(f(\alpha_2, \beta_2), g(\alpha_2, \beta_2)) \Rightarrow$$

$$f(\lambda(\alpha_2, \beta_2)) = \mu f(\alpha_2, \beta_2) \text{ and } g(\lambda(\alpha_2, \beta_2)) = \mu g(\alpha_2, \beta_2)$$

This leads us to considering **homogeneous polynomials**: $P \in \mathbb{C}[z, w]$ is homogeneous iff $\exists n \in \mathbb{N}$ such that $\forall \lambda, z, w \in \mathbb{C} : P(\lambda z, \lambda w) = \lambda^n P(z, w)$. In this case, we say that $P$ is a homogeneous polynomial of degree $n$.

Example for $n=3$: $P(z, w) = Az^3 + Bz^2w + Czw^2 + Dw^3$.

If we return to the diagram we have considered, we may choose $F$ to be
$F([z, 1]) = \left[\frac{P_1(z, 1)}{P_2(z, 1)}\right]$, where $P_1, P_2$ are homogenous polynomials of the same
degree $n$. That is, we may consider the mappings $\frac{A}{B}$, where $A, B \in \mathbb{C}[t]$ have
the same degree $n$.

Now, our aim is to make $F$ be a holomorphic mapping, in some sense.
For this, we need to prepare the setting in which we work, that is we want
to view $\mathbb{C}P^1$ as a **Riemann surface** (not a Riemann manifold). We say that $X$ is a n-dimensional topological manifold if $X$ is a topological space
locally homeomorphic to $\mathbb{R}^n$. Usually we want to make our life easier so we
will require that the manifold satisfies some additional properties as: it is
Hausdorff, connected, $\sigma$-compact ($X$ can be written as an ascending union
of compact sets, which will allow us to consider only a countable family of charts), paracompact.

We made several observations as for instance that a connected space is
not necessarily Hausdorff. For this we considered the real line from which
we deleted 0 and replaced it by 2 points. The topology changes in that the
neighborhoods of the 2 additional points become the neighborhoods of 0 from
which we delete zero and add the appropriate point. Such a space, remains
connected, but it is not Hausdorff because the 2 additional points cannot
be separated by disjoint neighborhoods. There was another example about
paracompact spaces, but I didn’t understand it.
Suppose that \((U, \varphi)\) and \((V, \psi)\) are 2 overlapping charts in a 2-dimensional manifold. If \(\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)\) is holomorphic for any 2 overlapping coordinate charts \((U, \varphi)\) and \((V, \psi)\), then \(X\) is called a **Riemann surface**.

We say that a continuous mapping \(f\) between 2 Riemann surfaces \(X\) and \(Y\) is holomorphic if no matter how we choose a point \(p \in X\) and a chart \((U, \varphi)\) around \(p\) and \((V, \psi)\) around \(f(p)\), then \(\psi \circ f \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)\) is holomorphic.

An important, amazingly ‘simple’ result that we mentioned is the **Uniformization theorem**: If \(X\) is a simply connected Riemann surface then \(X\) is biholomorphic to \(D(0,1), \mathbb{C}\) or \(\mathbb{C}P^1\).

Now, we come back to \(\mathbb{C}P^1\) viewed as \(S^2\) and cover it by 2 charts - the projection from the North pole \((0,0,1)\) and from the South pole \((0,0,-1)\). This is a way to define \(\mathbb{C}P^1\) as a Riemann surface.

We want to determine \(\text{Aut}(\mathbb{C}P^1) = \{f : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1 : f\text{ is 1-1, onto and holomorphic }\}\). We look at the \(f\)'s which stabilizes \(\infty \Rightarrow f(\infty) = \infty \Rightarrow f\) restricted to \(\mathbb{C}_z\) is a proper holomorphic mapping. Because \(f\) is also bijective, from what we have proved before (we remember \(\text{Aut}(\mathbb{C})\)), it follows that \(f\) has the form \(f(z) = az + b, a \neq 0\). Now we consider the transitive part of \(\text{Aut}(\mathbb{C}P^1)\), that is the \(f\)'s with the property \(f(\infty) = z_0 \in \mathbb{C}\). If we compose such an \(f\) with \(\varphi(z) = \frac{1}{z - z_0} (z_0 \not\to \infty)\), we obtain a mapping which fixes \(\infty\) (a mapping from the stabilizer of \(\text{Aut}(\mathbb{C}P^1)\)). This helps us to show that \(f\) has the form \(f(z) = \frac{az + b}{cz + d}\), where \((a,b), (c,d)\) are linearly independent.

What we have shown (almost):

\[
\text{Aut}(\mathbb{C}P^1) = \left\{ \frac{az + b}{cz + d} : \det \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \neq 0 \right\}
\]

It can be shown that \(\text{Aut}(\mathbb{C}P^1)\) is a group and its elements are called linear fractional transformations, Möbius transformations, etc. It is also denoted \(\text{PGL}_2((\mathbb{C})\) and contains \(SO(3)\)(rotations of the unit sphere), \(SU(1,1)\) (automorphisms of the unit disc), \(\text{Aff}(\mathbb{C}) = \{az + b : a \neq 0\}\)(automorphisms of the complex plane).