Homework #7  Math 503 November 29, 2004

Due Monday, December 13, 2004

The following three problems are from Remmert’s Theory of Complex Functions.

Problem 1: For $a > 1$ show that $\int_{0}^{2\pi} \frac{d\phi}{a + \sin \phi} = \frac{2\pi}{\sqrt{a^2 - 1}}$.

Problem 2: Prove the identity $\int_{-\infty}^{\infty} dx \frac{dx}{(x^2 + a^2)^2} = \frac{3\sqrt{2}\pi}{8a^7}$ for $a > 0$.

Problem 3: Prove that $\int_{0}^{\infty} \frac{\sqrt{x}}{x^2 + a^2} dx = \frac{\pi}{\sqrt{2a}}$.

The following problem is Exercise 239 in $\mathbb{N}^2$. Similar problems are found on written qualifying exams of many universities.

Problem 4: For each positive integer $n$, and for each real $\lambda > 1$, prove that the equation $z^n = e^{z - \lambda}$ has no solutions with $|z| = 1$, and exactly $n$ simple solutions with $|z| < 1$.

A continuous function is proper if and only if the inverse image of every compact set is compact. The following problem is Exercise 297 in $\mathbb{N}^2$.

Problem 5: Prove that there is no proper holomorphic map from the open unit disc into the complex plane.

The following problem is from Conway’s Functions of One Complex Variable.

Problem 6: Does there exist a holomorphic function $f : D(0, 1) \rightarrow D(0, 1)$ with $f\left(\frac{1}{2}\right) = \frac{3}{4}$ and $f'\left(\frac{1}{2}\right) = \frac{2}{3}$?

The following problem is from Remmert’s Theory of Complex Functions. Here $H$ is the open upper halfplane, so $H = \{ z \in \mathbb{C} : \text{Im } z > 0 \}$.

Problem 7: Let $f : D(0, 1) \rightarrow H$ be holomorphic with $f(0) = i$. If $f(0) = i$, prove that
a) $\frac{1 - |z|}{1 + |z|} \leq |f(z)| \leq \frac{1 + |z|}{1 - |z|}$ for $z \in D(0, 1)$.
b) $|f'(0)| \leq 2$.

Problem 8: Let $f$ be a holomorphic function which maps the unit disk into the unit disk. Show that $|f(z) + f(-z)| \leq 2|z|^2$ for all $z$ in the unit disk, and if the equality holds for some $z$, then $f(z) = e^{i\theta}z^2$ for some real $\theta$.
This semester I’ve worked with a study group of grad students who are preparing for our written exams. We looked at this problem. At the urging of the very kind students in the study group, I advise you that the problem statement, copied directly from the Johns Hopkins exam, is incorrect. Please do one of the following two alternative problems.

# 1 Find a counterexample to the problem as stated. Then add a simple hypothesis to the problem which makes it correct, and solve the resulting problem.

# 2 Solve this problem, quoted from Remmert’s *Theory of Complex Functions*. Here $E$ is the unit disc.

Let $f : E \to E$ be holomorphic, with $f(0) = 0$. Let $n \in \mathbb{N}$, $n \geq 1$, $\zeta := e^{2\pi i/n}$. Show that

\[
(*) \quad |f(\zeta z) + f(\zeta^2 z) + \cdots + f(\zeta^n z)| \leq n|z|^n \quad \text{for all } z \in E.
\]

Moreover, if there is at least one $c \in E \setminus \{0\}$ such that equality prevails in (*) at $z = c$, then there exists an $a \in \partial E$ such that $f(z) = az^n$ for all $z \in E$.

*Hint.* Consider the function $h(z) := \frac{1}{nz^{n-1}} \sum_{j=1}^{n} f(\zeta^j z)$. For the proof of the implication $f(\zeta z) + f(\zeta^2 z) + \cdots + f(\zeta^n z) = naz^n \Rightarrow f(z) = az^n$, verify that the function $k(z) := f(z) - az^n$ satisfies

\[
k(\zeta z) + k(\zeta^2 z) + \cdots + k(\zeta^n z) = 0 \quad \text{and} \quad |az^n| + 2\Re(az^n k(\zeta^j z)) + |k((\zeta^j z)|^2 < 1
\]

for every $j \in \{0, 1, \ldots, n-1\}$, and consequently $|k(z)|^2 < n(1 - |z|^{2n})$. 

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