Problem 1: Suppose \( f \) is holomorphic in the unit disc, and 
\[
 f\left(\frac{\sqrt{-1}}{k}\right) = \frac{100}{k^4}
\]
for integer \( k \geq 2 \). What is \( f \) exactly?

Problem 2: Let \( f \) be a function defined in a domain \( D \) in \( \mathbb{C} \) such that \( f(z)^3 \) is holomorphic in \( D \).

a) Is \( f(z) \) holomorphic in \( D \)? (Prove or give a counterexample to your answer.)

b) If \( f \in C^1(D) \), can you conclude that \( f \) is holomorphic in \( D \)? (Verify your answer.)

Problem 3: Without using the Fundamental Theorem of Algebra, prove for any \( P \) of degree \( n \geq 1 \) that
\[
 \lim_{R \to \infty} \int_{\partial D(0,R)} \frac{P'(z)}{P(z)} \, dz = 2\pi i \cdot n.
\]
Deduce the Fundamental Theorem of Algebra from this equality.

Problem 4: Suppose that \( U \) is an open subset of \( \mathbb{C} \), and \( \{K_n\}_{n \in \mathbb{N}} \) is a sequence of compact sets so that \( \bigcup_{n \in \mathbb{N}} K_n = U \) and \( K_n \subseteq \text{interior}(K_{n+1}) \) for all \( n \in \mathbb{N} \). Let \( C(U) \) be the vector space of continuous complex-valued functions defined on \( U \). If \( f \) and \( g \) are in \( C(U) \), define \( d(f, g) \) by
\[
 d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}}
\]
where \( \|h\|_S = \sup\{ |h(z)| : z \in S \} \). Prove that \( (C(U), d) \) is a complete metric space, and that \( f_j \xrightarrow{d} f \) if and only if \( \{f_j\} \) converges uniformly to \( f \) on every compact subset of \( U \).

Problem 5: Find all uniformly continuous entire functions.

Problem 6: Suppose that \( f \) and \( g \) are entire functions satisfying \( |f(z)| \leq |g(z)| \) for all \( z \in \mathbb{C} \). Prove that \( f(z) = Cg(z) \) for some \( C \in \mathbb{C} \).

Problem 7: Suppose in this problem that \( f \) has an isolated singularity at 0.

a) Prove that if \( e^f \) has a removable singularity at 0, then \( f \) must have a removable singularity at 0.

b) Under what conditions does \( e^f \) have a pole at 0?

\( \heartsuit \) This is how the question was phrased. I think the sentence is weird.