Homework #2  Math 503  September 13, 2004
Due Wednesday, September 22, 2004

Please read §1.2 (pp. 10–22) in N².

Problem 1: Suppose that \{a_n\} and \{b_n\} are sequences of non-negative real numbers and \{c_n\} is the sequence defined by \(c_n = a_n b_n\). Let \(A = \limsup a_n\), \(B = \limsup b_n\) and \(C = \limsup c_n\). If \{a_n\} converges, prove that \(C = AB\). Show by example that if we do not assume \{a_n\} converges, then \(C\) and \(AB\) may not be equal. What relationship must hold between \(C\) and \(AB\) whether or not convergence is assumed, and why?

Problem 2: Show that the series
\[
\frac{1}{1 + |z|} - \frac{1}{2 + |z|} + \frac{1}{3 + |z|} - \frac{1}{4 + |z|} + \cdots + \frac{(-1)^{n-1}}{n + |z|} + \cdots
\]
is not absolutely convergent but is uniformly convergent in the whole complex plane.

From Classical Complex Analysis by Liang-sin Hahn and Bernard Epstein.

Problem 3: Suppose that \(\varphi(r)\) is a function defined for \(r \geq 0\) which is bounded in every finite interval and tends to \(\infty\) as \(r \to \infty\). Prove that there is an entire function*, \(F\), which is real on \(\mathbb{R}\) so that \(F(r) \geq \varphi(r)\) for all real \(r \geq 0\).

**Hint** You may consider an everywhere convergent power series \(\sum_{n=1}^{\infty} \left(\frac{z}{n}\right)^{\lambda_n}\) for a “sufficiently rapidly increasing sequence of positive integers”, \(\{\lambda_n\}\).

From Analytic Functions by Stanislaw Saks and Antoni Zygmund, and attributed to Poincaré.

Problem 4: Prove that if \(|z| < 1\) then
\[
\sum_{n=1}^{\infty} \frac{z^n}{1 - z^n} = \sum_{m=1}^{\infty} d(m) z^m
\]
where \(d(m)\) is the number of divisors of the positive integer \(m\). Also, prove that both series converge uniformly on compact subsets of \(D(0,1)\), the unit disc.

The generating function of \(\{d(m)\}\), Sequence A000005 in Sloane’s On-Line Encyclopedia of Integer Sequences.

Problem 5: Show that the power series
\[
\frac{z^3}{1} - \frac{z^2}{1} + \frac{z^3}{2} - \frac{z^2}{2} + \cdots + \frac{z^3}{n} - \frac{z^2}{n} + \cdots
\]
has radius of convergence 1, and that the points of convergence and those of divergence of this series each form sets which are everywhere dense in \(\partial D(0,1)\).

**Hint** Take points of the form \(z = \exp\left(\frac{\pi i k}{3^N}\right)\) and consider the case of \(k\) odd and \(k\) even.

From Analytic Functions by Stanislaw Saks and Antoni Zygmund, and attributed to Vijayaraghavan.

* A function which is holomorphic in all of \(\mathbb{C}\) is called *entire*.

OVER
Problem 6: Do Exercise 48 of $N^2$, which follows.

For each function $f : \Omega \to \mathbb{C}$ holomorphic on a connected open set $\Omega \subseteq \mathbb{C}$ prove the following statements.

(48.1) If $f'(z) = 0$ for every $z \in \Omega$, then $f$ is constant.
(48.2) If there exists $c \in \mathbb{C}$ such that $f(z) = c \cdot \overline{f(z)}$ for every $z \in \Omega$, then $f$ is constant.
(48.3) If $f(\Omega) \subseteq \mathbb{R}$, then $f$ is constant.
(48.4) If $|f|$ is constant, then $f$ is constant.
(48.5) If $g : \mathbb{C} \to \mathbb{C}$ is holomorphic and $g \circ f$ is constant, then $f$ or $g$ is constant.
(48.6) If $f_1, f_2, \ldots, f_N$ are holomorphic in $\Omega$ and if $|f_1|^2 + |f_2|^2 + \cdots + |f_N|^2$ is constant, then each $f_j$ is constant.