An example of two-dimensional heat flow

I want steady-state solutions for the two-dimensional heat equation, \( u_{tt} = u_{xx} + u_{yy} \), in one the square which has sides parallel to the coordinate axes and each side \( \pi \) units long, with lower-left hand corner is at the origin, \((0,0)\). Since \( u \) is supposed to be a steady-state solution, \( u_t = 0 \) always, and we can omit the \( t \) in the variables we give \( u \). We are actually looking for solutions \( u(x,y) \) to Laplace’s equation, \( u_{xx} + u_{yy} = 0 \) in the \( \pi \)-by-\( \pi \) square. The boundary conditions are:

\[
(BC) \quad u(x,0) = 0 \text{ & } u(x,\pi) = 0 \text{ for } 0 \leq x \leq \pi; \quad u(0,y) = 0 \text{ & } u(\pi,y) = 1 \text{ for } 0 \leq y \leq \pi
\]

Here are Maple commands to generate a partial sum of the Fourier sine series for the function 1 (a function which is always 1):

\[
> c := n \rightarrow (2/Pi) \cdot \int (1 \cdot \sin(n \cdot y), y=0..Pi); \\
> \text{plot} (\text{sum}(c(j) \cdot \sin(j \cdot y), j=1..50), y=0..Pi, \text{thickness}=2, \text{color}=\text{black});
\]

As can be expected, the graph is all fuzzy at the ends (Gibb’s phenomenon again). Now we can try to look at a partial sum of the solution to Laplace’s equation:

\[
> u := (x,y) \rightarrow \text{sum}((1/\sinh(j \cdot Pi)) \cdot c(j) \cdot \sin(j \cdot y) \cdot \sinh(j \cdot x), j=1..50); \\
\]

Maple reports that \( u(1,2) \) is 1.176183537. We can draw some pictures with these commands:

\[
> \text{plot3d} (u(x,y), x=0..Pi, y=0..Pi, \text{axes}=\text{normal}); \\
> \text{contourplot} (u(x,y), x=0..Pi, y=0..Pi, \text{contours}=10, \text{color}=\text{black});
\]

Below to the left is a picture of the surface \( z = u(x,y) \). On the right is a contour plot of \( u(x,y) \):

Here are some slices of this surface by planes perpendicular to the \( xy \)-plane. The commands were:

\[
> \text{plot}([u(x,1), u(x,3), u(x,5)], x=0..Pi, \text{color}=\text{black}, \text{thickness}=2); \\
> \text{plot}([u(1,y), u(3,y), u(7,y)], y=0..Pi, \text{color}=\text{black}, \text{thickness}=2);
\]

Which slices are which curves?