Answers to the Second Exam 11/20/2004

(12) 1. Complete the definitions. a) Suppose $v_1, v_2, \ldots$ and $v_t$ are vectors in $\mathbb{R}^n$. Then $v_1, v_2, \ldots$ and $v_t \text{ are linearly independent if whenever } \sum_{j=1}^t c_j v_j = 0, \text{ all } c_j \text{ must be 0.}$ b) Suppose $A$ is an $n \times n$ matrix. $\lambda$ is an eigenvalue of $A$ if there is a non-zero vector $X$ in $\mathbb{R}^n$ so that (writing $X$ as a column vector) $AX = \lambda X$ or if $\lambda$ is a root of det$(A - \lambda I_n) = 0$.

(22) 2. Suppose that $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Note $A$ is not symmetric! a) Compute the characteristic polynomial of $A$. Answer det$(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 0 & 1 \\ 0 & 1 - \lambda & 0 \\ 1 & 1 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^3 - (1 - \lambda) = (1 - \lambda)(1 - \lambda)^2(2 + \lambda)$.

b) Find the eigenvalues of $A$. Answer 0 and 1 and 2.

c) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$. Answer Solve linear systems for each $\lambda$: $(A - \lambda I)X = 0. 0: (1 1 1) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ so } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; 1: (0 0 1) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ so } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.

d) Find a diagonal matrix $D$ and an invertible matrix $P$ so that $P^{-1}AP = D$. Answer $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

e) Find $P^{-1}$. Answer $\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ . f) Compute $Z = AP$. Answer $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

g) Compute $P^{-1}Z$ using the results of d) and e). Answer $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. 

h) Write $A$ as a product of $D$ and $P$ and $P^{-1}$ (in the correct order!) and use this information to compute $A^6$. Note The entries in the answer are 0, 1, 31, or 32. Answer Since $P^{-1}AP = D$, $A = PD P^{-1}$ so that $A^6 = PD^6 P^{-1}$. And $D^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{pmatrix}$ so $PD^6 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 64 \\ 0 & -1 & 0 \\ 0 & 0 & 64 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 64 \\ 0 & -1 & 0 \\ 0 & 0 & 64 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 32 & 31 & 32 \\ 32 & 31 & 32 \\ 32 & 32 & 32 \end{pmatrix}$.

(12) 6. Suppose $f(x) = 3\sin(2x) - 5\cos(3x) + 2\cos(4x)$. Which of these integrals is larger: $\int_{-\pi}^{\pi} (f(x))^2 \, dx$ or $\int_{-\pi}^{\pi} (f'(x))^2 \, dx$? Answer Orthogonality implies $\int_{-\pi}^{\pi} (f(x))^2 \, dx = \pi(9 + 25 + 4) = 38\pi$. $f'(x) = 6\sin(2x) - 15\cos(3x) + 8\cos(4x)$, so $\int_{-\pi}^{\pi} (f'(x))^2 \, dx = \pi(36 + 225 + 64) = 325\pi$. The derivative integral is larger.

(8) 7. Suppose $f(x) = x + x^4$ for $x \in [0, \pi]$. a) If $F(x)$ is the odd extension of $f(x)$ to $[-\pi, \pi]$, use the formula for $f(x)$ to write a simple formula for $F(x)$. Be sure you specify $F(x)$ for all $x$‘s in $[-\pi, \pi]$. Answer Since $F(-x) = -f(x)$, we see specifications can be $F(x) = x + x^4$ for $x \in [0, \pi]$, and $F(x) = -(x - (\pi) + (\pi)^4) = x - x^4$ for $x$ in $[\pi, 0]$. Which terms must be 0 in the Fourier series of $F(x)$? Answer All of the Fourier cosine terms (alternatively, the Fourier sine coefficients must be 0). b) If $G(x)$ is the even extension of $f(x)$ to $[-\pi, \pi]$, use the formula for $f(x)$ to write a simple formula for $G(x)$. Be sure you specify $G(x)$ for all $x$‘s in $[-\pi, \pi]$. Answer Since $G(-x) = f(x)$, we see that specifications can be $G(x) = x + x^4$ for $x \in [0, \pi]$, and $G(x) = -(x - (\pi) + (\pi)^4) = x + x^4$ for $x$ in $[-\pi, 0]$. Which terms must be 0 in the Fourier series of $G(x)$? Answer All of the Fourier sine terms (alternatively, the Fourier cosine coefficients must be 0).
3. In this problem the functions \( f(x) \) and \( g(x) \) and \( h(x) \) are piecewise linear functions. Parts of their graphs are shown to the right. The domains of these functions are all real numbers (all of \( \mathbb{R} \)). The functions are 0 where the graphs are not shown. a) Prove that the functions \( f(x) \) and \( g(x) \) and \( h(x) \) are linearly independent.

**Answer** If \( Af(x) +Bg(x) +Ch(x) = 0 \) then we can try some special values of \( x \) and get a system of linear equations:
\[
\begin{align*}
A = 0 & \quad \text{if } x = 1; \\
A + B = 0 & \quad \text{if } x = 2; \\
A + B + C = 0 & \quad \text{if } x = 3.
\end{align*}
\]
Therefore the only linear combination of \( f(x) \) and \( g(x) \) and \( h(x) \) which is 0 is the trivial linear combination and the functions are linearly independent.

b) The function \( Q(x) \) is piecewise linear and part of its graph is shown to the right. The domain of \( Q(x) \) is all real numbers (all of \( \mathbb{R} \)) and the function \( Q(x) \) is 0 where the graph is not shown. Can \( Q(x) \) be written as a linear combination of \( f(x) \) and \( g(x) \) and \( h(x) \)?

**Answer** If \( Q(x) = Af(x) +Bg(x) +Ch(x) \) then we can try some special values of \( x \) and get a system of linear equations:
\[
\begin{align*}
A = 2 & \quad \text{if } x = 1; \\
A + B = 0 & \quad \text{if } x = 2; \\
A + B + C = 0 & \quad \text{if } x = 3.
\end{align*}
\]
The first equation shows that \( A \) must be 2, then the second equation gives \( B = -2 \) and the third yields \( C = 0. \) But the fourth equation gives \( B = 2 \) which is a contradiction. There is no solution. \( Q(x) \) cannot be written as a linear combination of \( f(x), g(x), \) and \( h(x). \)

4. Suppose \( M \) is the matrix \[
\begin{pmatrix}
a + b & 0 & 0 \\
b & -1 & 0 \\
c & 2 & -1 \\
b - c & 1 & -1
\end{pmatrix}
\]
Prove that \( M \) is not invertible exactly when the vector \((a, b, c)\) in \( \mathbb{R}^3 \) is perpendicular to the vector \((1, 1, -2)\) in \( \mathbb{R}^3. \)

**Answer** We first compute \( \det M \) by expanding along the first row. So \( \det M = (a+b) \det \begin{pmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} - 1 \det \begin{pmatrix} b & -1 & 0 \\ c & 2 & -1 \\ b - c & 1 & -1 \end{pmatrix}. \) This is \((a+b)(-2+1) - 1(b(-2+1) - (-1)(-c + b - c)) = -(a+b) - (-b - c + b - c) = -a - b + 2c. \) \( M \) is not invertible when this is 0, which is the same as requiring that the dot product of \((a, b, c)\) and \((1, 1, -2)\) is 0.

5. In this problem, \( f(x) = x + 1. \) a) Compute \( \int f(x) \sin(nx) \, dx. \) **Answer** Integrate by parts:
\[
u = x + 1 \\
dv = \sin(nx) \, dx \\
v = \int -\frac{1}{n} \cos(nx) \, dx
\]
The integral we want is \((x + 1)\left(-\frac{1}{n} \cos(nx)\right) - \int -\frac{1}{n} \cos(nx) \, dx = (x + 1) \left(-\frac{1}{n} \cos(nx)\right) + \frac{1}{n^2} \sin(nx) + C.\)

b) Compute \( b_n = \int_0^\pi f(x) \sin(nx) \, dx \) as explicitly as you can when \( n \) is a positive integer.

**Answer** \((x + 1)\left(-\frac{1}{n} \cos(nx)\right) + \frac{1}{n^2} \sin(nx)\) \(\bigg|_{x=0}^{x=\pi} = (\pi + 1) \left(-\frac{1}{n}\right) (-1)^n - (-\frac{1}{n}) = (-1)^{n+1} \left(\pi + \frac{n+1}{n}\right) + \frac{1}{n}.\)

c) Give exact values for \( b_1 \) and \( b_2 \) and \( b_3 \) and \( b_4. \)

**Answer** \( b_1 = \pi + 2; \) \( b_2 = -\frac{\pi}{4}; \) \( b_3 = \frac{\pi}{4} + \frac{\pi}{6}; b_4 = -\frac{\pi}{12}. \)

d) Suppose \( g(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} b_n \sin(nx) \) and \( h(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} b_n \sin(nx). \) **Below** are two graphs of \( f(x) = x + 1 \) for \( x \) in \([0, \pi].\) Sketch a reasonable approximation to \( g(x) \) on the left graph. Sketch a reasonable approximation to \( h(x) \) on the right graph.

\[
\text{Graph of } g(x), \text{ the } 100^{\text{th}} \text{ partial sum of the Fourier sine series on } [0, \pi]
\]

\[
\text{Graph of } h(x), \text{ the sum of the whole Fourier sine series on } [0, \pi]
\]