13) 1. Suppose \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) are real sequences, and that for all positive integers, \( n \), \( x_n \leq y_n \leq z_n \). If both \( \{x_n\} \) and \( \{z_n\} \) converge and have the same limit, \( L \), prove that \( \{y_n\} \) converges and its limit is \( L \). Answer Fix \( \varepsilon > 0 \). Since \( \lim_{n \to \infty} x_n = L \) there is \( N_x \in \mathbb{N} \) so that if \( n \geq N_x \), then \( |x_n - L| < \varepsilon \). Therefore for such \( n \), \( L - \varepsilon < x_n < L + \varepsilon \). Similarly, there is \( N_z \in \mathbb{N} \) so that if \( n \geq N_z \), then \( L - \varepsilon < z_n < L + \varepsilon \). Let \( N = \max(N_x, N_z) \). If \( n \geq N \), then \( L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon \), so that \( |y_n - L| < \varepsilon \). Thus \( \lim_{n \to \infty} y_n = L \).

13) 2. Suppose \((X, d)\) is a metric space. If \( P \) and \( Q \) are connected subsets of \( X \) with \( P \cap Q \neq \emptyset \), prove that \( P \cup Q \) is connected. Answer Suppose there is a separation, \( A \) and \( B \), of \( P \cup Q \). Then \( A \cup B = P \cup Q \), \( \overline{A} \cap B = \emptyset \), \( A \cap \overline{B} = \emptyset \), and neither \( A \) nor \( B \) is empty. Since \( P \cap Q \neq \emptyset \), there is \( x \in P \cap Q \) so that \( x \in A \) or \( x \in B \). We address the first case (the second is similar). Consider \( A \cap P \) and \( B \cap P \). Since \( \overline{A \cap P} \subset \overline{A \cap P} \subset \overline{A} \) and \( B \cap P \subset B \) and therefore \( (A \cap P) \cap (B \cap P) = \emptyset \) (and similarly reversing the roles of \( A \) and \( B \)). But \( x \in A \), so since \( P \) is connected, \( P \subset A \). Similarly, \( Q \subset A \). Therefore \( B = \emptyset \) which is a contradiction, so no separation exists, and \( P \cup Q \) is connected.

15) 3. Suppose \((X, d)\) is a metric space.
   a) If \( A \) and \( B \) are subsets of \( X \), prove that \( \overline{A \cup B} = \overline{A} \cup \overline{B} \). Answer If \( x \in \overline{A} \), then given \( r > 0 \), \( N_r(x) \cap A \neq \emptyset \). That is, either \( x \in A \) or \( x \) is a limit point of \( A \). Since \( A \subset A \cup B \), if \( x \in \overline{A} \), then given \( r > 0 \), \( N_r(x) \cap (A \cup B) \neq \emptyset \) and thus \( x \in \overline{A \cup B} \). The case \( x \in \overline{B} \) is similar. Now if \( x \in \overline{A \cup B} \), consider \( N_r(x) \cap (A \cup B) = (N_r(x) \cap A) \cup (N_r(x) \cap B) \). This is not empty because \( x \) is an element of the closure of \( A \cup B \). If there is \( r > 0 \) so that \( N_r(x) \cap A = \emptyset \), there is always \( b \in B \) with \( b \in N_r(x) \). Also, if \( 0 < s < r \), there must be \( b \in B \) with \( b \in N_s(x) \) or else \( N_s(x) \cap (A \cup B) = \emptyset \). Therefore \( x \in \overline{B} \). The situation if \( N_r(x) \cap B = \emptyset \) is similar, so either \( x \in \overline{A} \) or \( x \in \overline{B} \).
   b) Give an example to show that the closure of the union of a countable number of subsets of \( X \) need not be equal to the union of the closures of each of the sets. Answer Take \( X = \mathbb{R} \) with the usual metric, and \( A_j = \{1/j\} \) for positive integer \( j \). Here \( \overline{A_j} = A_j \) but \( 0 \in \bigcup_{j=1}^{\infty} A_j \). So the union of the closures is not the same as the closure of the union.
   c) Give an example to show that \( \overline{A \cap B} \) and \( \overline{A} \cap \overline{B} \) need not be equal. Here \( A \) and \( B \) are subsets of \( X \). Answer Take \( X = \mathbb{R} \) with the usual metric. Suppose \( A = [0,1) \) and \( B = [1,2] \) so that \( \overline{A} = [0,1] \) and \( \overline{B} = [1,2] \). Therefore \( A \cap B = \emptyset \) so the closure is empty, but \( \overline{A} \cap \overline{B} = \{1\} \).

15) 4. Suppose \((X, d)\) is a metric space.
   a) If \( A \) is a subset of \( X \), prove that \( \text{diam}(A) = \text{diam}(\overline{A}) \). Comment \( \text{diam}(S) = \sup \{ d(x,y) : x, y \in S \} \) if \( S \subset X \). Answer Since \( A \subset \overline{A} \), the sup for \( \overline{A} \) is taken over more real numbers, and therefore \( \text{diam}(A) \leq \text{diam}(\overline{A}) \). If \( \text{diam}(A) < \text{diam}(\overline{A}) \), then there is \( \delta > 0 \) so that \( \text{diam}(A) + \delta < \text{diam}(\overline{A}) \) and therefore \( d(x,y) + \delta < \text{diam}(\overline{A}) \) for all \( x \) and \( y \) in \( A \). But the diameter of the closure is a sup, so there must be \( z \) and \( w \) in \( \overline{A} \) so that \( d(x,y) + \delta/2 < d(z,w) \) for all \( x \) and \( y \) in \( A \). Since \( z \in \overline{A} \) and \( w \in \overline{A} \), there are elements \( \hat{z} \) and \( \hat{w} \) in \( A \) with \( d(z,\hat{z}) < \delta/4 \) and \( d(w,\hat{w}) < \delta/4 \). Estimate:
   \[ d(z,w) \leq d(z,\hat{z}) + d(\hat{z},\hat{w}) + d(\hat{w},w) < d(\hat{z},\hat{w}) + 2(\delta/4) = d(\hat{z},\hat{w}) + \delta/2. \]
   This contradicts a
previous assertion (with \( \hat{z} \) as \( x \) and \( \hat{w} \) as \( y \)) so the diameters must be equal. (The text’s proof is more economical.)

b) Give an example of a subset \( A \) of \( X \) with \( \text{diam}(A) \neq \text{diam}(A^o) \) and \( A^o \neq \emptyset \). (\( A^o \) is the interior of \( A \).) \textbf{Answer} Take \( X = \mathbb{R} \) with the usual metric. If \( A = [0,1] \cup \{2\} \), then \( A^o = (0,1) \), \( \	ext{diam}(A) = 2 \), and \( \text{diam}(A^o) = 1 \).

(15) 5. a) Suppose \( (X,d) \) is a metric space, \( K \) is a compact subset of \( X \), \( U \) is an open subset of \( X \), and \( K \subset U \). Prove that there is \( r > 0 \) so that \( \bigcup_{k \in K} N_r(k) \subset U \). \textbf{Answer} Suppose \( k \in K \). Since \( U \) is open, there is \( r_k > 0 \) with \( N_{2r_k}(k) \subset U \). Then \( \{N_{r_k}(k)\}_{k \in K} \) is an open cover of \( K \) (with no 2 here!). \( K \) is compact so there is a finite subcover, \( \{N_{r_{k_j}}(k_j)\}_{1 \leq j \leq n} \).

Define \( r = \min\{r_{k_j} : 1 \leq j \leq n\} \). \( s \) is a positive real number since it is the minimum of a finite set of positive real numbers. If \( k \in K \), then there is \( k_j \) with \( d(k_j,k) < r \) (cover!). But \( N_r(k) \subset N_{2r}(k_j) \) (triangle inequality) and \( N_{2r}(k_j) \subset N_{2r_{k_j}}(k_j) \subset U \). So we have proved \( \bigcup_{k \in K} N_r(k) \subset U \).

\textbf{Alternative proof} Suppose no such \( r \) exists. Then for any positive integer \( n \) we can find \( k_n \in K \) and \( v_n \notin U \) with \( d(k_n,v_n) < \frac{1}{n} \). Since \( K \) is compact, the sequence \( \{k_n\} \) has a subsequence which converges to \( q \) in \( K \). But \( q \in U \) so there’s \( \delta > 0 \) with \( N_\delta(q) \subset U \). Find \( n \) so that \( \frac{1}{n} < \frac{\delta}{2} \) and \( d(k_n,q) < \frac{\delta}{2} \), possible since \( q \) is a subsequential limit of \( \{k_n\} \). Then (by \( \Delta \leq \delta \)) \( v_n \in N_{\frac{1}{n}}(k_n) \subset N_{\frac{\delta}{2}}(q) \subset N_\delta(q) \subset U \). But this contradicts \( v_n \notin U \).

b) Give an example to show that there can be a closed subset \( C \) of \( X \) and an open subset \( U \) of \( X \) with \( C \subset U \) so that there is no \( r > 0 \) with \( \bigcup_{x \in C} N_r(x) \subset U \). \textbf{Answer} Take \( \mathbb{R} \) with the usual metric and let \( C \) be the positive integers and \( U \) be the open set \( \bigcup_{n \in \mathbb{N}} \left( \left( \frac{1}{n}, \frac{n}{n+1} \right) \right) \). The Archimedean property implies there is no positive \( r \) with \( r < \frac{1}{n} \) for all \( n \in \mathbb{N} \), so this \( C \) is as desired. It is not difficult to find examples of \textit{connected} \( C \)'s and \( U \)'s satisfying this question in \( \mathbb{R}^2 \).

(14) 6. a) Prove directly from the definition of compactness that the half-open interval \( (0,1] \subset \mathbb{R} \) is not compact. (\( \mathbb{R} \) has the usual topology.) \textbf{Answer} Take \( U_n = \left( \frac{1}{n}, 1 \right] \). Then \( \{U_n\}_{n \in \mathbb{N}} \) is an open cover of \( (0,1] \) and \( U_{n+1} \supset U_n \). It is a cover by the Archimedean property. The cover “nests” since \( \frac{1}{n+1} < \frac{1}{n} \). If \( \{U_n\}_{1 \leq j \leq N} \) is a finite subcover, \( \bigcup_{1 \leq j \leq N} U_{n_j} = U_M \) where \( M = \max\{n_j : 1 \leq j \leq N\} \). But \( U_M = \left( \frac{1}{M}, 1 \right] \neq (0,1] \) by the Archimedean property.

b) Prove that a Cauchy sequence in a metric space is bounded. \textbf{Answer} Proved in class and in the text.

(15) 7. Suppose the following is known about three sequences:

If \( n \) is a positive integer, then \( |x_n - 2| < \frac{5}{n}, |y_n - 6| < \frac{20}{\sqrt{n}}, \) and \( |z_n - 5| < \frac{6}{n^2} \).

Then the sequences \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) converge, and their respective limits are 2, 6, and 5.

The sequence whose \( n \)th term is \( x_n y_n - z_n \) converges and its limit is \( 2 \cdot 6 - 5 = 7 \). Do not prove this, but find and verify a specific \( n \) so that \( |(x_n y_n - z_n) - 7| < \frac{1}{1,000} \). This need not be a “best possible” \( n \) but you must supply a specific \( n \) and a proof of your estimate. \textbf{Answer} \(|x_n y_n - z_n - 7| = |x_n y_n - z_n - (2 \cdot 6 - 5)| \leq |x_n y_n - 2y_n + 2y_n - 2 \cdot 6 + |z_n - 5| \leq |x_n - 2| |y_n| + |y_n - 6| + |z_n - 5| \). Suppose \( n \geq (100)^2 \). Then \( |y_n - 6| < \frac{20}{\sqrt{n}} = \frac{1}{100} \) so that \( |y_n| \leq |y_n - 6| + 6 < 7 \) as well as \( |y_n - 6| < \frac{20}{\sqrt{n}} \). Further, we know \( |x_n - 2| |y_n| < \frac{5}{n} \). Therefore \(|(x_n y_n - z_n) - 7| < \frac{35}{n} + \frac{20}{\sqrt{n}} + \frac{6}{n^2} \). Wow! Now take \( n = 10^{10} \) so \( \frac{35}{n} = \frac{35}{10^{10}} < \frac{1}{3,000}, \frac{20}{\sqrt{n}} = \frac{20}{10^{5}} < \frac{1}{3,000}, \) and \( \frac{6}{n^2} = \frac{6}{10^{20}} < \frac{1}{3,000} \). The total will be less than \( \frac{1}{1,000} \).