1

(20) 1. a) Suppose that a recursively defined sequence is given by $a_n = \begin{cases} 1, & \text{if } n = 1 \\ 2 + \sqrt[n]{a_{n-1}}, & \text{if } n > 1 \end{cases}$. Prove that $\{a_n\}$ converges and find its limit. Give supporting reasoning.

b) Suppose that a recursively defined sequence is given by $b_n = \begin{cases} 1, & \text{if } n = 1 \text{ or } 2 \\ b_{n-1} + \frac{1}{(b_{n-2})^2}, & \text{if } n > 2 \end{cases}$. Does $\{b_n\}$ converge? If not, why not? If it does, find its limit. Give supporting reasoning.

(20) 2. a) Suppose $(X, d)$ is a metric space, $S$ is a subset of $X$, and $w \in X$. Define “$w$ is a limit point of $S$”. Then consider $S$, a subset of $\mathbb{R}^2$, defined by $S = \{(x, y) \in \mathbb{R}^2 : x = \frac{1}{n} \text{ and } y = \frac{1}{m} \text{ for } n, m \text{ positive integers}\}$. Find all limit points of $S$ in $\mathbb{R}^2$ with the usual metric.

b) Suppose $(X, d)$ is a metric space, and $U$ and $C$ are subsets of $X$. Define “$U$ is an open set” and “$C$ is a closed set”. Prove that $C$ is closed if and only if every limit point of $C$ is in $C$.

(20) 3. Suppose $a < c < b$, $\alpha$ is an increasing function on $[a, b]$, and $f$ is a bounded real-valued function on $[a, b]$ which is continuous on $[a, c)$ and $(c, b]$. These assumptions hold for both parts of this problem.

a) Prove that if $\alpha$ is continuous at $c$, then $f \in \mathcal{R}(\alpha)$.

b) Find an example of an $\alpha$ which is not continuous at $c$ and an $f$ which is not continuous at $c$ so that $f \not\in \mathcal{R}(\alpha)$.

(20) 4. True or false. If true, give a very brief explanation of why the statement is correct. If false, supply an example showing why the implication is false.

a) Suppose $f: \mathbb{R} \to \mathbb{R}$ is uniformly continuous and bounded. Then $f$ is constant. **TRUE OR FALSE?**

b) Suppose a metric space has the property that every real-valued continuous function is bounded. Then the metric space is compact. **TRUE OR FALSE?**

c) Every metric space is the union of a collection of open balls of finite radius which are pairwise disjoint (the intersection of any two of the balls is empty). **TRUE OR FALSE?**

d) Every complete metric space is connected. **TRUE OR FALSE?**

e) If $\{a_n\}$ is a complex sequence for which $\sum_{n=1}^{\infty} |a_n|$ converges, then $\limsup_{n \to \infty} |a_n|^{1/n} < 1$. **TRUE OR FALSE?**

(20) 5. Suppose $S^1$ is the subset of $\mathbb{R}^2$ of points $(x, y)$ which satisfy the equation $x^2 + y^2 = 1$. Here $\mathbb{R}^2$ and $\mathbb{R}$ have the usual metrics. Prove that there is no 1-1 (injective) continuous map from $S^1$ to $\mathbb{R}^1$.

**Note** The mapping is not required to be onto (surjective).
6. Suppose \( \{a_n\} \) and \( \{b_n\} \) are real positive* sequences.
   a) Prove that \( \limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n \).
   b) Give an example to show that \( \limsup_{n \to \infty} (a_n + b_n) = \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n \) may be false.

7. Suppose \((X,d)\) is a metric space and \(S\) is a non-empty subset of \(X\). If \(w \in X\), define \(D(w) = \inf_{x \in S} d(x,w)\).
   a) Prove that \(D: X \to \mathbb{R}\) is continuous (it is actually uniformly continuous).
   b) Prove that \(D(x) = 0\) if and only if \(x \in \overline{S}\) (\(x\) is in the closure of \(S\)).
   c) If \(X = \mathbb{R}\) with the usual metric, is the function \(D\) differentiable everywhere for every choice of \(S\)?

8. Suppose \(f\) is a continuous real-valued function on \((0, \infty)\) and define \(F: (0, +\infty) \to \mathbb{R}\) by \(F(x) = \int_1^x f\left(\frac{u^2 + 1}{u}\right) \frac{du}{u}\). Prove that \(F\left(\frac{1}{x}\right) = -F(x)\) for all \(x > 0\).

9. Suppose \(f\) is continuous on \([0, 1]\) and \(\varepsilon > 0\). Prove that there is a piecewise linear function \(g\) on \([0, 1]\) so that \(|g(x) - f(x)| < \varepsilon\) for all \(x \in [0, 1]\).
   Advice: \(g\) is a piecewise linear function on \([0, 1]\) if \(g\) is continuous on \([0, 1]\) and there is a finite partition \(P = \{0 = x_0 < x_1 < \ldots < x_n = 1\}\) of \([0, 1]\) so that \(g|_{[x_j, x_{j+1}]}\) is equal to an affine function \(\langle A_j x + B_j \rangle\) for all integers \(j\) from 0 to \(n - 1\).

10. a) In this part of the problem, we consider sequences \(\{x_n\}\) in \(\mathbb{R}\) with the usual metric. Then give examples with brief explanation of:
    i) A sequence in \(\mathbb{R}\) which has an uncountable number of distinct subsequential limits.
    ii) A sequence in \(\mathbb{R}\) which has a countably infinite number of distinct subsequential limits.
    iii) A sequence in \(\mathbb{R}\) which has exactly three distinct subsequential limits.
    
   b) In this part of the problem, \(g: \mathbb{R} \to \mathbb{R}\) and \(g\) is differentiable at 2 with \(g'(2) = 4\). Show that there exists \(\delta > 0\) so that if \(2 < x < 2 + \delta\) then \(g(x) > g(2) + 3(x - 2)\).

* Word inserted after the exam!
Final Exam for Math 411

December 16, 2008

NAME _______________________________________

Do all problems, in any order.

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