1. Suppose \( \{x_n\}, \{y_n\}, \text{ and } \{z_n\} \) are real sequences, and that for all positive integers, \( n \), \( x_n \leq y_n \leq z_n \). If both \( \{x_n\} \) and \( \{z_n\} \) converge and have the same limit, \( L \), prove that \( \{y_n\} \) converges and its limit is \( L \).

2. Suppose \((X, d)\) is a metric space. If \( P \) and \( Q \) are connected subsets of \( X \) with \( P \cap Q \neq \emptyset \), prove that \( P \cup Q \) is connected.

3. Suppose \((X, d)\) is a metric space.
   a) If \( A \) and \( B \) are subsets of \( X \), prove that \( A \cup B = \overline{A \cup B} \).
   b) Give an example to show that the closure of the union of a countable number of subsets of \( X \) need not be equal to the union of the closures of each of the sets.
   c) Give an example to show that \( \overline{A} \cap B \) and \( A \cap \overline{B} \) need not be equal. Here \( A \) and \( B \) are subsets of \( X \).

4. Suppose \((X, d)\) is a metric space.
   a) If \( A \) is a subset of \( X \), prove that \( \text{diam}(A) = \text{diam}(\overline{A}) \).
      **Comment** \( \text{diam}(S) = \sup \{d(x, y) : x, y \in S\} \) if \( S \subset X \).
   b) Give an example of a subset \( A \) of \( X \) with \( \text{diam}(A) \neq \text{diam}(A^o) \) and \( A^o \neq \emptyset \). (\( A^o \) is the interior of \( A \).)

5. a) Suppose \((X, d)\) is a metric space, \( K \) is a compact subset of \( X \), \( U \) is an open subset of \( X \), and \( K \subset U \). Prove that there is \( r > 0 \) so that \( \bigcup_{k \in K} N_r(k) \subset U \).
   b) Give an example to show that there can be a closed subset \( C \) of \( X \) and an open subset \( U \) of \( X \) with \( C \subset U \) so that there is no \( r > 0 \) with \( \bigcup_{x \in C} N_r(x) \subset U \).

6. a) Prove directly from the definition of compactness that the half-open interval \((0, 1] \subset \mathbb{R} \) is not compact. (\( \mathbb{R} \) has the usual topology.)
   b) Prove that a Cauchy sequence in a metric space is bounded.

7. Suppose the following is known about three sequences:
   
   If \( n \) is a positive integer, then \( |x_n - 2| < \frac{5}{n}, |y_n - 6| < \frac{20}{\sqrt{n}} \), and \( |z_n - 5| < \frac{6}{n^2} \).
   Then the sequences \( \{x_n\}, \{y_n\}, \text{ and } \{z_n\} \) converge, and their respective limits are 2, 6, and 5. The sequence whose \( n^{th} \) term is \( x_n y_n - z_n \) converges and its limit is \( 2 \cdot 6 - 5 = 7 \). Do not prove this, but find and verify a specific \( n \) so that \( |(x_n y_n - z_n) - 7| < \frac{1}{1,000} \). This need not be a “best possible” \( n \) but you must supply a specific \( n \) and a proof of your estimate.
First Exam for Math 411

October 20, 2008

NAME ________________________________

Do all problems, in any order.

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