Sources of the 403:02 final exam problems

These problems are mildly edited from the qualifying exams of the universities indicated.

Oklahoma 1. Let \( u(x, y) = x^3 + x - 3xy^2 \).
   a) Show that \( u(x, y) \) is harmonic on the complex plane.
   b) Find all harmonic conjugates of \( u(x, y) \).
   c) Find an analytic function \( f(z) \) so that \( \text{Re } f = u \) and find the Taylor series of \( f(z) \) about the point 0.

Purdue 2. Construct a one-to-one analytic map from \( Q = \{ z : |z| < 1 \) and \( \text{Im } z > 0 \} \) (the upper half of the unit disc) onto the unit disc, \( U = \{ z : |z| < 1 \} \). Show how the boundary of \( Q \) is mapped to the boundary of \( U \).

Temple 3. Use the Residue Theorem to compute \( \int_{-\infty}^{\infty} \frac{1}{(x^2 + 2x + 2)^2} \, dx \).

Berkeley 4. Prove that for any fixed complex number \( \zeta \), \( \frac{1}{2\pi} \int_{0}^{2\pi} e^{2\zeta \cos \theta} \, d\theta = \sum_{n=0}^{\infty} \left( \frac{\zeta^n}{n!} \right)^2 \).
   Hint Use the “dictionary” to convert this into a line integral and then use infinite series.

Temple 5. Let \( \mathbb{R}^- = \{ x \text{ is real and } x \leq 0 \} \). Suppose \( f(z) \) is analytic in \( \mathbb{C} \setminus \mathbb{R}^- \), and \( f(x) = x^x \) for real positive \( x \). Find \( f(i) \) and \( f(-i) \).
   Scoring 10 points for the values, and 10 points for explanation.

Temple 6. Show that if \( f(z) \) is analytic at \( a \) and \( g(z) = \frac{f(z) + af'(a) - zf'(a) - f(a)}{(z - a)^2} \) then \( g(z) \) has a removable singularity at \( z = a \). What value should be given to \( g(a) \) so that the extended function is analytic at \( a \)?

Johns 7. Find the number of zeros of the function \( f(z) = 2z^5 + 8z - 1 \) in the annulus \( 1 < |z| < 2 \).

Hopkins

Missouri 8. Suppose \( |f(z)| \leq K \) on the circumference of a square whose side length is \( L \), and let \( z_0 \) be the center of the square. If \( f(z) \) is analytic in a domain containing the square, show that \( |f'(z_0)| \leq \frac{8K}{\pi L} \).
   Hint Use an integral formula.

Penn 9. Prove that if \( f(z) \) is an entire function and if there is a positive number \( M \) so that \( \text{Re } f(z) \leq M \) for all \( z \), then \( f(z) \) is constant.

Florida 10. Suppose the Bernoulli polynomials are defined by the Taylor expansion \( \frac{ze^{wz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(w)}{k!} z^k \). Find the first three Bernoulli polynomials, \( B_0(w) \), \( B_1(w) \), and \( B_2(w) \).