A Fourier series example

Math 403, section 2

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Just as power series are infinite linear combinations of monomials, \( \{x^n\}_{n=0}^{\infty} \), Fourier series (the specific series given here is called a Fourier sine series) are infinite linear combinations of the functions \( \{\sin nx\}_{n=0}^{\infty} \). Power series have very nice behavior inside their radius of convergence: “The sum is continuous inside the radius of convergence” and “The series can be differentiated ‘term-by-term’ inside the radius of convergence”. Fourier series generally aren’t as nice as power series, and this example is presented to explain why these properties of power series should not be accepted as clear.

The example is the series \( \sum_{n=1}^{\infty} \frac{\sin(n^4x)}{n^2} \) on the interval \([0, 2\pi] \). Since all the sines are \(2\pi\) periodic, this interval displays all of the behavior of the sum. Suppose \( N \) is a positive integer and \( F_N \) is defined by \( F_N(x) = \sum_{n=1}^{N} \frac{\sin(n^4x)}{n^2} \), the \( N \)th partial sum of the infinite series. From left to right below are the graphs of \( F_5, F_{10}, \) and \( F_{20} \) as drawn by Maple.

It certainly seems reasonable that these pictures are stabilizing, and tending to some complicated but continuous limit. Now here are Maple pictures of the derivatives of \( F_5, F_{10}, \) and \( F_{20} \).

Look at the the vertical axes. The derivatives are behaving even more wildly than a first glance would indicate. They are growing and wiggling enormously. Nothing is canceling. It seems reasonable that the sum of the derivatives would not converge.

\[ \implies \text{Power series behave very nicely; Fourier series may not.} \]