A triple integral
computed by iterated integrals two ways

The integrand ("mass density") is $5x + 7y + 11z$, and the region in $\mathbb{R}^3$ is the tetrahedron with vertices (corners) $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$, and $(0, 0, 0)$. I guessed at the equation for the tilted face of the tetrahedron: $x + \frac{1}{2}y + \frac{1}{3}z = 1$.

The mass is the triple integral over the region. So the mass is $\int \int \int_{\text{tetrahedron}} \text{density} \, dV$. We converted this into two different triply iterated integrals.

We first did the order $dx \, dy \, dz$. Here I went from “outside in”, and the limits on $z$ went from 0 to 3. An intermediate $z$-slice is shown in the middle picture. Then, going inside once more, the $y$ limits are 0 and $2(1 - x)$ (the intersection of the line in that picture with the $y$-axis). Finally, the $x$ limits go from 0 to the line, so $x = 1 - \frac{1}{2}y - \frac{1}{3}z$.

The result:

$$\text{Mass} = \int_{z=0}^{z=3} \int_{y=0}^{y=2(1-x)} \int_{x=0}^{x=1-\frac{1}{2}y-\frac{1}{3}z} (5x + 7y + 11z) \, dx \, dy \, dz.$$ A Maple computation of this is shown below. Look at $A$, $B$, and $C$.

Then we tried the order $dz \, dy \, dx$. Here I went from “outside in”, and the limits on $x$ went from 0 to 1. An intermediate $x$-slice is shown in the rightmost picture. Then, going inside once more, the $y$ limits are 0 and $2(1 - \frac{1}{3}z)$ (the intersection of the line in that picture with the $y$-axis). Finally, the $z$ limits go from 0 to the line, so $z = 3(1 - \frac{1}{2}y - x)$.

The result:

$$\text{Mass} = \int_{x=0}^{x=1} \int_{y=0}^{y=2(1-(1/3)z)} \int_{z=0}^{z=3(1-\frac{1}{2}y-\frac{1}{3}z)} (5x + 7y + 11z) \, dz \, dy \, dx.$$ A Maple computation of this is shown below. Look at $A_1$, $B_1$, and $C_1$.

These iterated integrals compute the mass in Maple.

```
> A:=int(5*x+7*y+11*z,z=0..3*(1-x-(1/2)*y));
A := 5*x*(3-3*x-3/2*y) + 7*y*(3-3*x-3/2*y) + 11/2*(3-3*x-3/2*y)^2
> B:=int(A,y=0..2*(1-x));
B := 5/8*(2-2*x)^3 + 7/8*(2-2*x)^2 + 1/2*(2-2*x) + 11/2*(2-2*x)
> C:=int(B,x=0..1);
C := 13
> A1:=int(5*x+7*y+11*z,z=0..1-(1/2)*y-(1/3)*z);
A1 := 5/2*(1-1/2*y-1/3*z)^2 + 7*y*(1-1/2*y-1/3*z) + 11*z*(1-1/2*y-1/3*z)
> B1:=int(A1,y=0..1-(1/3)*z));
B1 := -23/24*(2-2/3*z)^3 + 7/2*(2-2/3*z)^2 + 11/2*(2-2/3*z)
> C1:=int(B1,z=0..3); 
C1 := 13
```

The $B_1$ value was centered so nothing got “lost” over the page’s edge. From all this we learn that $13 = 13$. 