1. Find all critical points of each function. Describe (as well as you can) the type of each critical point. Explain your conclusions.
   a) \( f(x, y) = (x^2 + y^2) e^{x-y} \)
   b) \( g(x, y) = (y - x^2)^{600} \)

2. Suppose that \( F(x, y, z) = y^2 e^{(6x-yz)} \). Note that \( F(1, 2, 3) = 4 \).
   a) Find a unit vector in the direction is the largest directional derivative of \( F \) at \((1, 2, 3)\). Find the value of that directional derivative.
   b) Find an equation for the plane tangent to the surface \( y^2 e^{(6x-yz)} = 4 \) at the point \((1, 2, 3)\).
   c) Find parametric equations for a line normal to the surface \( y^2 e^{(6x-yz)} = 4 \) at the point \((1, 2, 3)\).

3. The point \( p = (-2, 1, 1) \) satisfies the equation \( z^3 + xy^2 z + 1 = 0 \). Suppose near the point \( p \) that \( z \) is defined implicitly by the equation as a differentiable function of \( x \) and \( y \).
   a) If \( x \) is changed from \(-2\) to \(-2.03\) and \( y \) is changed from \(1\) to \(1.04\), use linear approximation to describe the approximate change in \( z \).
   b) What is the value of \( \frac{\partial^2 z}{\partial x^2} \) at \( p \)?

4. Prove Green’s Theorem for the region in the plane bounded by the \( x \)-axis and the curve \( y = 1 - x^2 \) by explicitly computing both sides of the equality for a “general” \( P(x, y) \, dx + Q(x, y) \, dy \) (be sure to state what conditions on \( P \) and \( Q \) are needed) and checking that the two sides are indeed the same.

5. Suppose a vector field is defined by \( \mathbf{F} = (y^2 z) \mathbf{i} + (2xyz) \mathbf{j} + (xy^2 + 4z) \mathbf{k} \).
   a) Determine whether there is a scalar function \( P(x, y, z) \) defined everywhere in space such that \( \nabla P = \mathbf{F} \). If there is such a \( P \), find it; if there is not, explain why not.
   b) Compute the integral \( \int_W \mathbf{F} \cdot \mathbf{T} \, ds \), where \( W \) is the circular helix whose position vector is given by \( \mathbf{R}(t) = (\cos t) \mathbf{i} + (\sin t) \mathbf{j} + t \mathbf{k} \) for \( 0 \leq t \leq 2\pi \). Use information gotten from your answer to a) to help if you wish.

6. The average value of a function \( f \) defined in a region \( R \) of \( \mathbb{R}^3 \) is \( \frac{\iiint_R f \, dV}{\iiint_R 1 \, dV} \). Compute the average distance to the center of a sphere of radius \( a \).

7. Suppose \( f(x, y, z) = xy^2 z^3 \).
   a) Compute \( \int_0^1 \int_0^x \int_0^y f(x, y, z) \, dz \, dy \, dx \).
   b) Write the integral in a) as a sum of one or more iterated integrals in \( dx \, dy \, dz \) order. You are not asked to integrate your answer, only to set it up.
(20) 8. Sketch the three level curves of the function \( W(x, y) = ye^x \) which pass through the points \( P = (0, 2) \) and \( Q = (2, 0) \) and \( R = (1, -1) \). **Label each curve** with the **appropriate function value**. Be sure that your drawing is clear and unambiguous. Also, sketch on the same axes the vectors of the gradient vector field \( \nabla W \) at the points \( P \) and \( Q \) and \( R \) and \( S \) and \( T \). The point \( S = (0, -2) \) and the point \( T = (-2, 0) \).

(20) 9. Suppose \( \mathbf{F} = -2xz \mathbf{i} + y^2 \mathbf{k} \). **Note** There is no \( \mathbf{j} \) component in \( \mathbf{F} \).
   a) Compute curl \( \mathbf{F} \).
   b) Compute the outward unit normal \( \mathbf{n} \) for the sphere \( x^2 + y^2 + z^2 = a^2 \).
   c) If \( R \) is any region on the sphere \( x^2 + y^2 + z^2 = a^2 \), verify that \( \iint_R (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = 0 \).
   d) Suppose \( C \) is a simple closed curve on the sphere \( x^2 + y^2 + z^2 = a^2 \). Show that the value of the line integral \( \int_C -2xz \, dx + y^2 \, dz \) is 0.
   **Comment** Please don’t attempt a direct computation! Use c) and one of the big theorems.

(20) 10. a) Verify that the improper integral \( \int_0^1 x^{-3/2} \, dx \) does not converge.
   b) Suppose \( R \) is the (roughly) triangular-shaped region in \( \mathbb{R}^2 \) defined by \( y = x^2 \), \( y = 0 \), and \( x = 1 \). For which values of \( a \) and \( b \) does the integral \( \iint_R x^a y^b \, dA \) converge?
Very difficult
Final Exam for Math 291, section 1

December 22, 2006

NAME __________________________________________

Do all problems, in any order.
Show your work. An answer alone may not receive full credit.
No notes or calculators may be used on this exam.
A page with formulas will be supplied.

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