Here are detailed answers to version A. Brief answers to version B are at the end.

(12) 1. Theoretical results imply that $x + 3yz$ has a maximum and a minimum on the sphere $x^2 + y^2 + z^2 = 1$. Use Lagrange multipliers to find these maximum and minimum values.

**Answer** Suppose \( f(x, y, z) = x + 3yz \) and \( g(x, y, z) = x^2 + y^2 + z^2 \). Then \( \nabla f = (1, 3z, 3y) \) and \( \nabla g = (2x, 2y, 2z) \) so that the Lagrange multiplier equations are (including the constraint equation) \( 1 = \lambda(2x), 3z = \lambda(2y), 3y = \lambda(2z), \) and \( 1 = x^2 + y^2 + z^2 \). Now we solve these equations. The first equation immediately tells us that neither \( x \) nor \( \lambda \) can be 0. The second and third equations imply if \( y = 0 \) then \( z = 0 \) (and vice versa). If both \( y \) and \( z \) are 0 then \( 1 = x^2 + y^2 + z^2 \) shows that \( x = \pm 1 \) so the objective function \( x + 3yz \) is \( \pm 1 \). Can we do better (get larger and smaller values of the objective function)? If no variable is 0, the second and third equations can be rewritten as \( \frac{3z}{2y} = \lambda \) and \( \frac{3y}{2z} = \frac{3y}{2z} \) and \( z = y^2 \). Then \( \lambda \) must be \( \pm \frac{3}{2} \) itself since \( y = \pm z \). So \( 1 = \lambda(2x) \) implies that \( x = \pm \frac{1}{3} \) and \( 1 = x^2 + y^2 + z^2 \) gives \( 1 = \left( \frac{1}{3} \right)^2 + 2z^2 \) and \( z = \pm \sqrt{\frac{1}{3}} \). If \( \lambda > 0 \), then \( x > 0 \), and \( y \) and \( z \) have opposite signs. With all + signs, \( x + 3yz \) becomes \( \frac{1}{3} + 3\left( \frac{1}{3} \right) = \frac{4}{3} \). With – signs for \( x \) and \( y \) and + for \( z \) we get \( -\frac{4}{3} \). These are the actual minimum and maximum values. To the right is a picture of the ball together with the surface \( x + 3yz = \frac{5}{3} \). They do indeed seem to be tangent (and at two points, the other corresponding to minus signs on \( y \) and \( z \)) just as the Lagrange multiplier method suggests.

(12) 2. Suppose \( I = \int_0^3 \int_0^2 xy \ dy \ dx \).

a) Compute \( I \).

**Answer** \( \int_0^3 \int_0^2 xy \ dy \ dx = \int_0^3 \left[ \int_0^2 y \ dy \right] \ dx = \int_0^3 \left[ \frac{1}{2}y^2 \right]_0^2 \ dx = \frac{1}{2} \int_0^3 4 \ dx = \frac{3}{2} \left[ 4 \right]_0^3 = 3 \).

b) Use the axes to the right to sketch the region of integration for \( I \).

**Answer** Shown to the right.

c) Write \( I \) as a sum of one or more \( dx \ dy \) integrals. You do not need to compute the result!

**Answer** \( \int_0^3 \int_0^2 xy \ dy \ dx + \int_0^3 \int_0^2 y \ xy \ dy \ dx \).

(12) 3. The coordinates \( (x, y, z) \) of points in a solid object \( A \) in \( \mathbb{R}^3 \) satisfy the inequalities \( 0 \leq z \leq x - y^2 \) and \( 0 \leq x \leq 1 \). Compute the triple integral of 1 over the object \( A \). (This is the volume of \( A \).)

**Note** Four views of the object were given.

**Answer** A picture of a slice with \( z \) fixed is to the right, where \( 0 < z < 1 \). The volume is \( \int_0^1 \int_0^z y \ dy \ dz + \int_0^1 \int_0^{1-z} y \ dy \ dz = \int_0^1 \int_0^{1-z} 2y \ dy \ dz = -\frac{1}{16} \left[ (1-z)^2 \right]_z^0 = \frac{3}{8} \).

A picture of a slice with \( x \) fixed is to the left, where \( 0 < x < 1 \) so another answer is \( \int_0^1 \int_0^{\sqrt{x}} \int_0^{1-y^2} 1 \ dz \ dy \ dx \).

(12) 4. Compute \( \int_D e^{-x^2 - y^2} \ dA \) where \( D \) is the region in the plane which is inside the unit circle (the circle with center at \( (0, 0) \) and radius 1) and also inside the upper half plane (where \( y \geq 0 \).

**Answer** A picture of the region is to the right. It is friendly to polar coordinates. The integral is \( \int_0^\pi \int_0^1 e^{-r^2} r \ dr \ d\theta = \int_0^\pi -\frac{1}{2} e^{-r^2} \bigg|_{r=0}^1 \ d\theta = \int_0^\pi -\frac{1}{2} (e^{-1} + \frac{1}{2}) \ d\theta = \frac{1}{2} (1 - \frac{1}{e}) \).
5. Express in cylindrical coordinates and evaluate: \( \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{z^2+y^2}} z \, dy \, dz \, dx \).

**Answer**: \( z \, dz \, dy \, dx \) becomes \( rz \, dr \, d\theta \, dz \). The boundary \( y = \sqrt{1-x^2} \) along with the knowledge that \( x \) goes from 0 to 1 describes the part of the unit disc in the first quadrant (similar to the setup of the previous problem) because \( y = \sqrt{1-x^2} \) is part of \( x^2 + y^2 = 1 \), and \( \sqrt{\cdot} \) is always non-negative square root. Since the \( z \) boundary description involves \( r \), I will change the order from \( z \, dr \, d\theta \, dz \) to \( zr \, dr \, d\theta \). The triple integral becomes \( \int_0^1 \int_0^{\pi/2} \int_0^r zr \, dr \, d\theta \) \( dr \) = \( \int_0^1 \int_0^{\pi/2} \int_0^r zr \, d\theta \, dz \, dr \) = \( \int_0^1 \int_0^{\pi/2} \frac{1}{3} r^3 \theta \bigg|_{\theta=0}^{\pi/2} d\theta \) \( dr \) = \( \int_0^1 \frac{\pi}{3} r^3 \, dr \) \( = \frac{\pi}{3} = \frac{\pi}{3} \).

(12)

6. Use spherical coordinates to calculate the triple integral of \( f(x,y,z) = x^2 + y^2 + z^2 \) over the region \( 1 \leq x^2 + y^2 + z^2 \leq 4 \).

**Answer**: \( \rho^2 = x^2 + y^2 + z^2 \) so the region is just \( 1 \leq \rho \leq 2 \) with all \( \theta ' s (0 \leq \theta \leq 2\pi) \) and all \( \phi ' s (0 \leq \phi \leq \pi) \). The integrand in spherical coordinates is \( \rho^2 \). So the desired triple integral is \( \int_0^{2\pi} \int_0^{\pi} \int_0^2 \rho^4 \sin(\phi) \, d\rho \, d\phi \, d\theta \) = \( \int_0^{2\pi} \int_0^{\pi} \left( \int_0^2 \rho^4 \sin(\phi) \, d\rho \right) d\phi d\theta \) = \( \int_0^{2\pi} \int_0^{\pi} \left( \frac{2^5}{5} \sin(\phi) \right) d\phi d\theta \) = \( \int_0^{2\pi} \frac{2^5}{5} \theta \bigg|_{\phi=0}^{\pi} d\theta \) \( = \frac{128\pi}{5} \), although \( \frac{2^5-1}{5} \) is simpler.

(12)

7. This problem is about the transformation \( \begin{cases} x = e^{3u} \cos(2v) \\ y = e^{3u} \sin(2v) \end{cases} \).

a) Compute the Jacobian of this transformation. The result should be \( 6e^{6u} \) but you must show the details of the computation. **Answer**: We need \( \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det \begin{pmatrix} 3e^{3u} \cos(2v) & -2e^{3u} \sin(2v) \\ 3e^{3u} \sin(2v) & 2e^{3u} \cos(2v) \end{pmatrix} \) = \( (3e^{3u} \cos(2v))(2e^{3u} \cos(2v)) - (-2e^{3u} \sin(2v))(3e^{3u} \sin(2v)) = 6e^{6u} \cos(2v)^2 + 6e^{6u} \sin(2v)^2 = 6e^{6u} \left( \frac{\cos(2v)^2 + \sin(2v)^2}{2} \right) \), and this is \( 6e^{6u} \).

b) Suppose \( R \) is the region in the \( uv \)-plane determined by \( u = 0 \), \( u = \frac{1}{2} \), \( v = 0 \), and \( v = \frac{\pi}{2} \) as shown on the coordinate axes below and to the left. Sketch the image region using this transformation in the \( xy \)-plane below and to the right.

(16)

8. a) Compute \( \int_C x \, dx + y^2 \, dy \) if \( C \) is a quarter circle centered at \((0,0)\) from \((1,0)\) to \((0,1)\) followed by a line segment from \((0,1)\) to \((3,1)\). \( C \) is shown in a diagram to the right. You may need more than one integral!

**Answer**: From \((0,0)\) to \((1,0)\) use \( x = \cos t \) and \( y = \sin t \) so \( dx = -\sin t \, dt \), \( dy = \cos t \, dt \), and \( 0 \leq t \leq \frac{\pi}{2} \). The integral over that portion of the curve is \( \int_0^{\pi/2} -\cos(t)(\sin(t)) + (\sin(t)^2)(\cos(t)) \, dt \) = \( -\frac{\sin(t)^2}{2} + \frac{\sin(t)^4}{4} \bigg|_0^{\pi/2} = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4} \). For the line segment, \( x = t \) and \( y = 1 \) so \( dx = dt \) and \( dy = 0 \, dt \), and \( 0 \leq t \leq 3 \). So this integral is \( \int_0^1 dt = \frac{9}{2} \). The total integral is therefore \( \frac{9}{2} - \frac{1}{4} = \frac{13}{3} \).

**Another method**: \( \varphi(x,y) = \frac{x^2}{2} + \frac{y^4}{8} \) is a potential for \( x \mathbf{i} + y^2 \mathbf{j} \) (verify this by checking \( \varphi \) using partial differentiation). Then the integral is \( \varphi(\text{The end}) - \varphi(\text{The start}) = \varphi(3,1) - \varphi(1,0) = \left( \frac{9}{2} + \frac{1}{8} \right) - \left( \frac{1}{2} + \frac{1}{8} \right) = \frac{13}{3} \).

b) Suppose \( \mathbf{F} \) is the vector field \((x+5y^2)\mathbf{i} + (Ax+y)\mathbf{j}\) where \( A \) is a constant. There is one value of \( A \) for which this vector field is a gradient vector field. Find that value of \( A \). Then find all potentials of \( \mathbf{F} \), using that value of \( A \). **Answer**: \( \frac{\partial}{\partial y} \) of \( x + 5y^2 \) is 10y, and \( \frac{\partial}{\partial x} \) of \( Ax + y \) is \( A \), so the desired value of \( A \) is 10.

Now \( \int x + 5y^2 \, dx = \frac{x^2}{2} + 5xy^2 + C_1(y) \) and \( \int 10xy \, dy = 5xy^2 + C_2(x) \) where \( C_1(y) \) and \( C_2(x) \) are unknown functions. But inspection of the two descriptions of the potential tells me that the most general potential of \( \mathbf{F} \) is \( \frac{x^2}{2} + 5xy^2 + C \) for any constant \( C \).

**Brief answers to version B**

1. \( x + 5y \) has answer \( \frac{1}{2}x^2 \). 2. \( I = \int_0^1 \int_0^3 xy \, dy \, dx = 65 \). The first partial integration has answer \( \frac{1}{2}x^2 - \frac{1}{2} \). The graph is similar, and \( c \)’s answer is \( \int_0^1 \int_0^3 xy \, dy \, dx + \int_0^3 \int_0^y xy \, dy \, dx \). The same. 4. \( \int_0^2 \int_0^1 e^{-r^2} r \, dr \, d\theta \) with the same answer. 5. The same. 6. The answer is \( \int_0^{\sqrt{3}-1} = \frac{18\sqrt{3}}{5} \). 7. A similar computation gives the stated answer. The graph in b) is the same. 8. a) Much the same parameterizations can be used. The answer is \( \frac{1}{6} - \frac{1}{6} = \frac{1}{6} + \frac{1}{6} \). In b), \( A=6 \) and the potential is \( \frac{1}{6} + 3xy^2 + C \) for any constant \( C \).