Formula sheet for the second exam in Math 251:05-10, spring 2006

The distance from \( P_0(x_0,y_0,z_0) \) to \( P_1(x_1,y_1,z_1) \) is \( \sqrt{(x_0-x_1)^2 + (y_0-y_1)^2 + (z_0-z_1)^2} \).

The distance from \( P_1(x_1,y_1,z_1) \) to plane \( ax+by+cz+d=0 \) is \( \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} \).

Sphere: \( (x-h)^2 + (y-k)^2 + (z-l)^2 = r^2 \).

Plane: \( a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \) where \( \mathbf{n} = \langle a, b, c \rangle \).

\[
\begin{align*}
x &= x_0 + at \\
y &= y_0 + bt \quad \text{through} \quad (x_0, y_0, z_0) \quad \text{direction} \quad \langle a, b, c \rangle.
\end{align*}
\]

\[
|a| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2} \quad \text{if} \quad a = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}.
\]

\[
a \cdot b = |a| |b| \cos \theta \quad (\text{If} \quad 0, \quad \text{then} \quad a \perp b.)
\]

\[
a \times b = -b \times a \quad a \cdot (b \times c) = (a \times b) \cdot c \quad a \times (b \times c) = (a \cdot c) b - (a \cdot b) c
\]

\[
\text{comp}_a b = \frac{a \cdot b}{|a|^2} \quad \text{proj}_a b = \frac{a \cdot b}{a \cdot a} \quad a \cdot (b \times c).
\]

Volume of a parallelepiped with edges \( \mathbf{a}, \mathbf{b}, \mathbf{c} \): \( |a \cdot (b \times c)| \)

Arc length: \( \int_a^b |r'(t)| \ dt; \quad \frac{ds}{dt} = |r'(t)|; \quad \mathbf{T}(t) = \frac{r'(t)}{|r'(t)|}; \quad \mathbf{N}(t) = \frac{T'(t)}{|T'(t)|}; \quad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)\)

\[
\kappa = \left| \frac{dT}{ds} \right| = \frac{|T'(t)|}{|r'(t)|} = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} \quad 2 \quad \text{dim} \quad 2 \quad \text{dim} \quad \left| y''(t)x'(t) - x''(t)y'(t) \right| \quad y = f(x) \quad \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}
\]

\[
\tau = \frac{(r'(t) \times r''(t)) \cdot r'''(t)}{|r'(t) \times r''(t)|^2} \quad \text{Frenet-Serret} \quad \frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}, \quad \frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B}, \quad \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}.
\]

Tangent plane to \( z = f(x,y) \) at \( P_0(x_0,y_0,z_0) \): \( z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \)

Linear approximation to \( f(x,y) \) at \( (a,b) \): \( f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \)

Tangent plane to \( F(x,y,z) = 0 \):

\[
F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0
\]

If \( y \) implicitly defined by \( y = f(x) \) in \( F(x,y) = 0 \) then \( \frac{dy}{dx} = -\frac{F_x}{F_y} \).

If \( z \) implicitly defined by \( z = f(x,y) \) in \( F(x,y,z) = 0 \) then \( \frac{dz}{dx} = -\frac{F_x}{F_z} \) and \( \frac{dz}{dy} = -\frac{F_y}{F_z} \).

\[
\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \quad D_uf(x,y,z) = \nabla f(x,y,z) \cdot u
\]

Some chain rules:

If \( z = f(x,y) \) and \( x = x(t) \) and \( y = y(t) \), then \( \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \).

If \( z = f(x,y) \) and \( x = g(s,t) \) and \( y = h(s,t) \), then \( \frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial h}{\partial s} \).

Suppose \( f_x(a,b) = 0 \) and \( f_y(a,b) = 0 \). Let \( H = H(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 \).

a) If \( H > 0 \) and \( f_{xx}(a,b) > 0 \), then \( f(a,b) \) is a local minimum.

b) If \( H > 0 \) and \( f_{xx}(a,b) < 0 \), then \( f(a,b) \) is a local maximum.

c) If \( H < 0 \), then \( f(a,b) \) is not a local maximum or minimum (\( f \) has a saddle point).
Lagrange multipliers for one constraint

If $G$ (the variables) = a constant is the constraint and we want to extremize the objective function, $F$ (the variables), then the extreme values can be found among $F$’s values of the solutions to the system of equations $\nabla G = \lambda \nabla F$ (a vector abbreviation for the equations $\lambda \frac{\partial F}{\partial x} = \frac{\partial G}{\partial x}$ where $*$ is each of the variables) and the constraint equation.

**Polar coordinates**

$x = r \cos \theta$  \hspace{1cm}  $y = r \sin \theta$  \hspace{1cm}  $r^2 = x^2 + y^2$  \hspace{1cm}  $\theta = \arctan \left( \frac{y}{x} \right)$

$dA = r \, dr \, d\theta$

**Spherical coordinates**

$x = \rho \sin \phi \cos \theta$  \hspace{1cm}  $y = \rho \sin \phi \sin \theta$  \hspace{1cm}  $z = \rho \cos \phi$

$\rho^2 = x^2 + y^2 + z^2$

$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

**Total mass** of a mass distribution $\rho(x, y, z)$ over a region $R$ of $\mathbb{R}^3$ is $\int \int \int_R \rho(x, y, z) \, dV$.

**Line integral formulas**

\[
\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{ \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 } \, dt
\]

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_C \mathbf{F} \cdot \mathbf{T} \, ds
\]

\[
\int_C P(x, y) \, dx + Q(x, y) \, dy = \int_a^b P(x(t), y(t)) x'(t) \, dt + Q(x(t), y(t)) y'(t) \, dt
\]

**Green’s Theorem**

\[
\int_C P \, dx + Q \, dy = \int \int_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]

These $P, Q$ pairs will give $R$’s area \( \left\{ \begin{array}{ll}
P=-y \text{ and } Q=0 & \text{ } \\
P=0 \text{ and } Q=x & \\
P=-\frac{1}{2}y \text{ and } Q=\frac{1}{2}x & \end{array} \right. \)

A conservative vector field $\mathbf{V} = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$ is a **gradient vector field**: there’s $f(x, y)$ with $\nabla f = \mathbf{V}$ so $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$. $f$ is a **potential** for $\mathbf{V}$. A conservative vector field is **path independent**. Work done by such a vector field over a **closed curve** is 0. For $\mathbf{V}$ conservative with potential $f$: $\int_C P \, dx + Q \, dy = f(\text{the end}) - f(\text{the start})$.

If $P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$ is conservative, then $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. If the region is **simply connected** (means no holes) then the converse is true, and $f$ is both $\int P(x, y) \, dx$ and $\int Q(x, y) \, dy$. 