Interpolating factorials

Here's a simple integral which we can compute and then complicate:

$$\int_0^A e^{-x}dx = -e^{-x}\Big]_0^A = -e^{-A} - (-e^0) = 1 - e^{-A}$$

If we let $A \to \infty$, then $e^{-A} \to 0$ so the improper integral $\int_0^\infty e^{-x}dx$ converges, and it has value 1. We'll use this in combination with a reduction formula. If n is a positive integer consider:

$$\int_0^A x^n e^{-x}dx = -x^n e^{-x}\Big]_0^A + n\int_0^A x^{n-1}e^{-x}dx$$

$$\int u\,dv \quad = \quad uv \quad - \quad \int v\,du$$

$$\left.\begin{matrix} u = x^n \\ dv = e^{-x}dx \end{matrix}\right\}\left\{\begin{matrix} du = nx^{n-1}dx \\ v = -e^{-x} \end{matrix}\right.$$

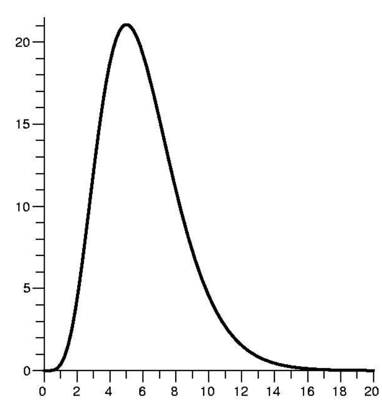
Again two $-$'s make a $+$: the $-$ from v and the $-$ from the integration by parts formula create a $+$ in our formula. Since n is a positive integer, both $x^n e^{-x}$ and $x^{n-1}e^{-x}$ are continuous in $[0, A]$. There is no "singularity" at 0. The lower limit of the $]$ term contributes nothing, since $0^n e^{-0}$ is 0 if n is a positive integer.

What happens as $A \to \infty$? The upper limit of the $]$ term gives $-A^n e^{-A}$. This has limit 0 since the exponential function grows faster than *any* power (just apply l'Hôpital's Rule n times).

If $\int_0^\infty x^{n-1}e^{-x}dx$ exists, then this argument shows that $\int_0^\infty x^n e^{-x}dx$ exists, and that $\int_0^\infty x^n e^{-x}dx = n\int_0^\infty x^{n-1}e^{-x}dx$. We know the value of this integral when $n = 0$ by the computation above. For larger integer n we now see

$$n! = \int_0^\infty x^n e^{-x}dx$$

when n is a positive integer.



$x^5 e^{-x}$ in the interval $[0, 20]$

Some simple facts about $x^n e^{-x}$ on $[0,\infty)$: it is always non-negative and is 0 only at 0. Its limit as $x \to \infty$ is 0. It has one critical point, a maximum, at $x = n$ where its value is $\left(\frac{n}{e}\right)^n$. It has two inflection points located at $n \pm \sqrt{n}$.

If n is positive, then the integral $\int_0^\infty x^n e^{-x}\,dx$ converges because when $x \geq 1$ we can compare it to the integral of a higher integer power which we already know converges. If we agree to make the following **definition**:

$$\text{If } n \geq 0 \text{ then } n! = \int_0^\infty x^n e^{-x}\,dx$$

then we first observe that this definition agrees with the usual one when n is a positive integer. Also, the integration by parts argument applies to show that $n! = n \cdot (n-1)!$ when $n \geq 1$.

One specific factorial value is

$$\left(\tfrac{1}{2}\right)! = \int_0^\infty \sqrt{x}\, e^{-x}\,dx$$

What is this number? We can try to approximate it by splitting the integral up into an infinite "tail" which is small and standard definite integral which can be approximated numerically.

$$\int_0^\infty \sqrt{x}\, e^{-x}\,dx = \int_0^{20} \sqrt{x}\, e^{-x}\,dx + \int_{20}^\infty \sqrt{x}\, e^{-x}\,dx$$

The infinite tail can be overestimated, and then the larger integral can be explicitly computed with one use of the reduction formula (the steps involving limits evaluating improper integrals are omitted):

$$\int_{20}^\infty \sqrt{x}\, e^{-x}\,dx < \int_{20}^\infty x e^{-x}\,dx = -xe^{-x} - e^{-x}\Big]_{20}^\infty = \frac{21}{e^{20}} \approx 4.238 \cdot 10^{-8}$$

Dropping the infinite tail will give an error which won't affect the first seven decimal places of the whole integral. Approximation by `Maple` tell us that $\int_0^{20} \sqrt{x} e^{-x}\,dx$ is about .8862269 so therefore

$$\int_0^\infty \sqrt{x} e^{-x}\,dx \approx .8862269$$

But $(.8862269)^2 \cdot 4 = 3.141592473$ is interestingly near π. If what this suggests is true then we should believe that $\left(\frac{1}{2}\right)! = \left(\frac{1}{2}\right) \cdot \sqrt{\pi}$ as was declared in the ball computation. I'd like to prove this now.

An obvious preliminary substitution makes $\int_0^\infty \sqrt{x} e^{-x}\,dx$ slightly easier to work with: $x = s^2$ so $dx = 2s\,ds$. The integral then becomes $\int_0^\infty 2s^2\, e^{-s^2}\,ds$. Another application of integration by parts will turn this into one of the most famous integrals in mathematics. I'll be careful, and regard the improper integral as a limit of the proper integrals below:

$$\int_0^A 2s^2\, e^{-s^2}\,ds = s \cdot -e^{-s^2}\Big]_0^A + \int_0^A e^{-s^2}\,ds$$

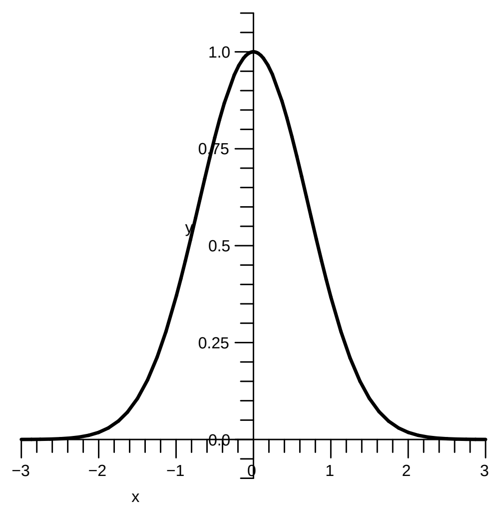
$$\int u\,dv \quad = \quad uv \quad - \quad \int v\,du$$

$$\left.\begin{array}{l} u = s \\ dv = e^{-s^2} 2s\, ds \end{array}\right\}\left\{\begin{array}{l} du = ds \\ v = -e^{-s^2} \end{array}\right.$$

The] term is 0 when $s = 0$ and has limit 0 as $A \to \infty$ since e^{-A^2} dies off much more quickly than A grows. Our original integral has now been changed to $\int_0^\infty e^{-s^2} ds$. We'll double it using the evenness of s^2 and instead try to show that

$$\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}$$

The function e^{-s^2} arises everywhere in probability and statistics. The presence of so many π's in statistical formulas is because of the value of this integral. Here's a picture of this famous "bell-shaped" curve:

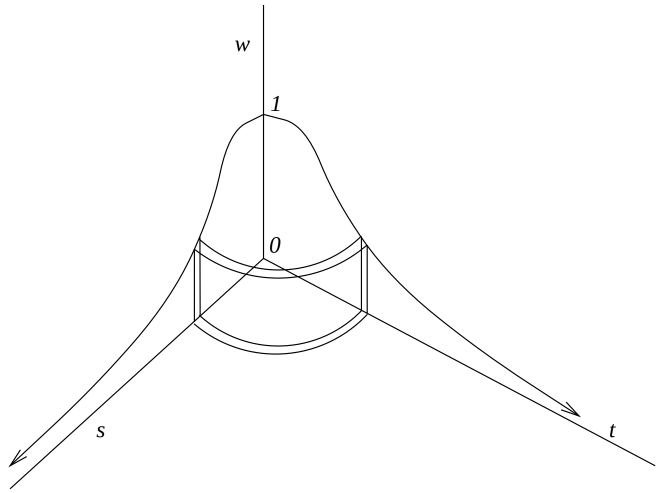


It has a maximum at 0 and inflection points at $\pm\frac{1}{\sqrt{2}}$.

There are many ways to conclude that $I = \int_{-\infty}^{\infty} e^{-s^2} ds$ is $\sqrt{\pi}$. Here is one of them.

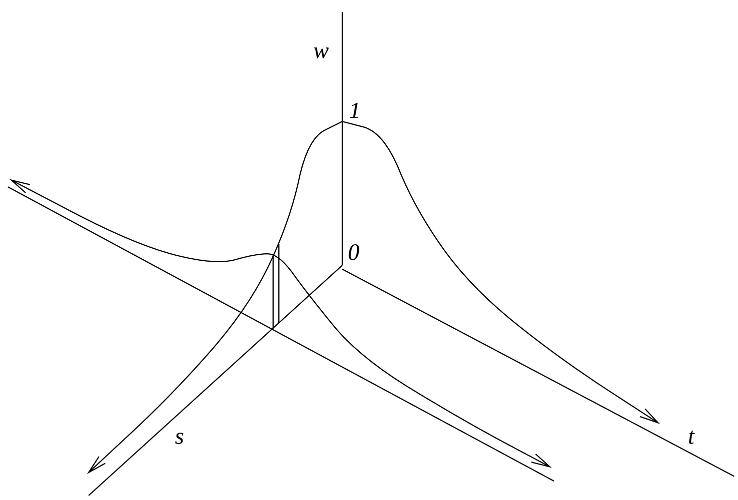
Revolve the curve $w = e^{-s^2}$ for $s \geq 0$ about the $w-$axis. The resulting solid of revolution has a volume, V, which can be computed with "thin shells".

$$V = \int_0^\infty 2\pi s e^{-s^2} ds = 2\pi \cdot -\frac{1}{2} e^{-s^2}\Big]_0^\infty = \pi$$

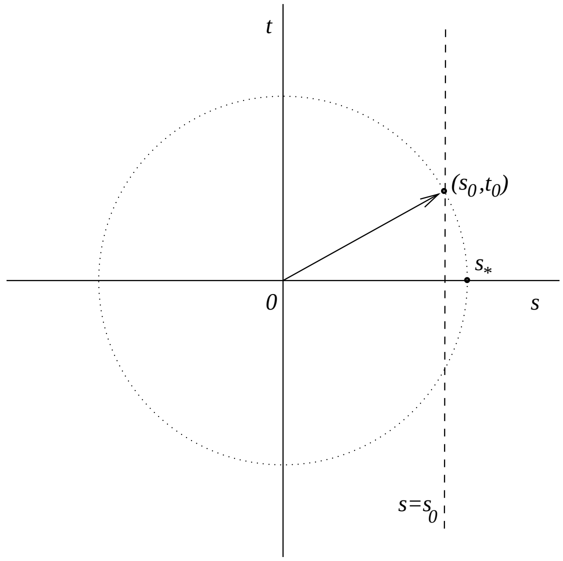


A quarter of the revolved curve, with a vertical strip ds thick forming a thin shell

Now let's compute V another way. We can slice the solid by planes perpendicular to the $s-$axis. We'll get a profile curve which will vary with s. A piece of the volume will be obtained by multiplying that profile curve by ds. What *is* the profile curve? A picture may help.



The profile curve looks a great deal like the original bell curve rescaled. This is correct, as the following more precise analysis shows. What's the height of the revolved curve over a point (s_0, t_0)? The height depends only on the distance of the point to the origin because the surface is composed of circles centered at $(0, 0)$. The height at (s_0, t_0) is the same height as the curve $w = e^{-s^2}$ has at $s_* = \sqrt{(s_0)^2 + (t_0)^2}$. Here is a picture looking *down* on the surface.



So the height of the surface at (s_0, t_0) is $e^{-\left(\sqrt{(s_0)^2+(t_0)^2}\right)^2} = e^{-(s_0)^2-(t_0)^2}$. The curve over the line $s = s_0$ has height over the point (s_0, t) the value $e^{-(s_0)^2-t^2} = e^{-(s_0)^2} \cdot e^{-t^2}$. Therefore the profile curve is the original curve rescaled: it is multiplied by $e^{-(s_0)^2}$. The area must be I, the original integral we want to compute, multiplied by $e^{-(s_0)^2}$. Let's move from the fixed cross-section at s_0 to any cross-section. The volume slice we want, which is the cross-sectional area times ds, will be $I \cdot e^{-s^2} ds$. We'll need to add these pieces of volume up from $s = -\infty$ to $s = \infty$. Therefore

$$V = \int_{-\infty}^{\infty} I \cdot e^{-s^2} ds = I \cdot \int_{-\infty}^{\infty} e^{-s^2} ds = I^2$$

Since we already know that $V = \pi$, we've confirmed that $I = \sqrt{\pi}$.

This rather involved demonstration of the value of I is an adaption of another way of computing I, using double integrals and polar coordinates. This method and others rely on the multiply/add property of the exponential function: $e^a \cdot e^b = e^{a+b}$. One of my colleagues says the tricks involved in evaluating this integral are almost "criminal acts" because they are so special and so clever.

History and notation This integral interpolation for factorials was discovered and investigated long ago. The resulting function was named the Gamma function, and the definition was given a shift in its argument. The Gamma function, $\Gamma(n)$ is defined by

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$$

so that $\Gamma(n) = (n-1)!$ when n is a positive integer. Please argue with Leonhard Euler (1707–1783) if you're unhappy about the Gamma function.