640:192:03 **Answers to Exam #1** **10/10/96**

(14) 1. Use the method of partial fractions to verify that

$$\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx = \frac{1}{4}\ln 2 + \frac{1}{8}\pi$$

Answer Write $\dfrac{1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1} = \dfrac{A(x^2+1)+(Bx+C)}{(x+1)(x^2+1)}$, so that $1 = A(x^2+1)+(Bx+C)(x+1)$. When $x=-1$ we see that $A=\frac{1}{2}$. Comparing x^2 coefficients of both sides, we see that $B=-1/2$. Finally, comparing constant coefficients on both sides, we see that $1 = A + C$ so $C = 1/2$.

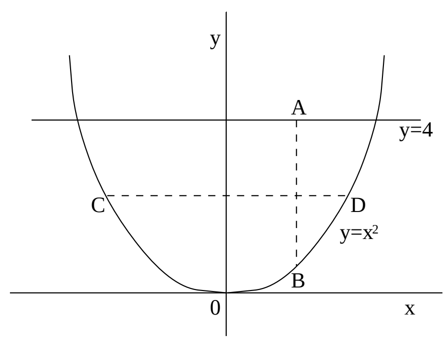
Now compute: $\displaystyle\int_0^1 \frac{1}{(x+1)(x^2+1)}\,dx = \int_0^1 \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}\,dx = \int_0^1 \frac{\frac{1}{2}}{x+1} + \frac{-\frac{1}{2}x}{x^2+1} + \frac{\frac{1}{2}}{x^2+1}\,dx =$
$\displaystyle\frac{1}{2}\ln(x+1) + -\frac{1}{4}\ln(x^2+1) + \frac{1}{2}\arctan(x)\Big]_0^1 = \Big(\frac{1}{2}\ln(2) + -\frac{1}{4}\ln(2) + \frac{1}{2}\arctan(1)\Big) -$
$\displaystyle -\Big(\frac{1}{2}\ln(1) + -\frac{1}{4}\ln(1) + \frac{1}{2}\arctan(0)\Big) = \frac{1}{4}\ln(2) + \frac{1}{2}\frac{\pi}{4} = \frac{1}{4}\ln 2 + \frac{\pi}{8}$. Whew! All done.

(16) 2. a) The base of a solid is the region bounded by the parabola $y = x^2$ and the line $y = 4$. Cross-sections of the solid by planes perpendicular to the $x-$axis are squares. Find the volume of the solid.

Answer The diagram shows the parabola and line. The segment $\overline{AB}$ is supposed to be the side of a square. The length of $\overline{AB}$ is $4 - x^2$, so the area of the square is $\left(4-x^2\right)^2$ and we get the volume by integrating the cross-sectional area (note that the bounds for the integral are obtained by solving $x^2 = y = 4$): $\displaystyle\int_{-2}^{2} 16 - 8x^2 + x^4\,dx = 16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\Big]_{-2}^{2} = \left(32 - \frac{64}{3} + \frac{32}{5}\right) - \left(-32 + \frac{64}{3} + \frac{32}{5}\right) = \frac{512}{15}$. I don't think the final step is necessary. I did it lazily with `maple`!

b) The base of a solid is the region bounded by the parabola $y = x^2$ and the line $y = 4$. Cross-sections of the solid by planes perpendicular to the $y-$axis are squares. Find the volume of the solid.

Answer Now the other way. $\overline{CD}$ is the side of the square. Here I think integration with respect to y is most natural. The curve bounding $\overline{CD}$ is $y = x^2$ which is also $x = \pm\sqrt{y}$ and the length of $\overline{CD}$ is $2\sqrt{y}$. The cross-sectional area is $\left(2\sqrt{y}\right) = 4y$ and the volume desired is $\displaystyle\int_0^4 4y\,dy = 2y^2\Big]_0^4 = 32$.



(12) 3. Use integration by parts followed by a substitution (or a substitution followed by integration by parts) to verify that

$$\int_0^1 e^{\sqrt{x}}\,dx = 2$$

Answer If we try the substitution first, the integral might be more friendly. So I'll try $w = \sqrt{x}$ and therefore $w^2 = x$ with $2w\,dw = dx$. The indefinite integral changes: $\displaystyle\int e^{\sqrt{x}}\,dx = \int e^w 2w\,dw$. The transformed integral is a well-known candidate for integration by parts. Let's do it (I'll save the 2 until later):

$$\int e^w w\,dw = we^w - \int e^w\,dw = we^w - e^w + C$$

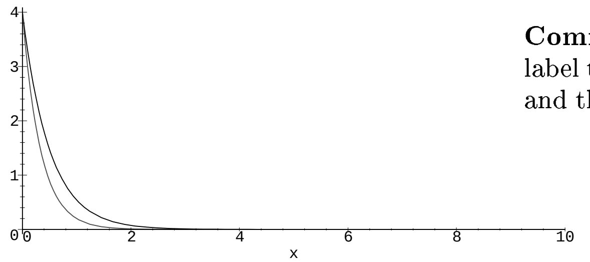
$$\int u\,dv \quad = \quad uv \quad - \quad \int v\,du$$

$$\left.\begin{array}{r} u = w \\ dv = e^w\,dw \end{array}\right\}\left\{\begin{array}{l} du = dw \\ v = e^w \end{array}\right.$$

We can reverse the substitution so the indefinite integral becomes (with the 2!) $2\left(\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}\right) + C$.

Therefore, $\int_0^1 e^{\sqrt{x}}\,dx = 2\left(\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}}\right)\Big]_0^1 = (2(e-e)) - (2\,(0-1)) = 2$.

(16) 4. Suppose $\mathcal{R}$ is the region in the first quadrant bounded by the two curves $y = 4e^{-2x}$ and $y = 4e^{-3x}$.
a) Sketch that part of the region between $x = 0$ and $x = 10$ on the axes given.
Answer Here's `maple`'s version (not "on the axes given"):



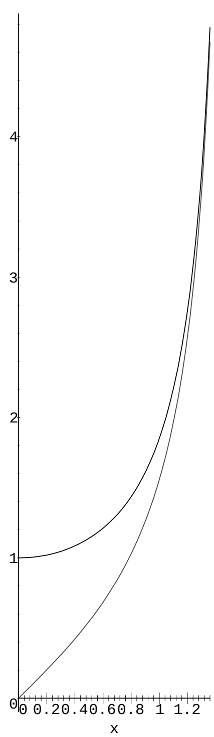
Comment I should have added a request to label the boundary curves in both this problem and the next. Pictures should have labels.

b) Compute the area of the whole region $\mathcal{R}$ (out to ∞) if it is finite.
Answer It certainly *looks* finite! (Of course, I said repeatedly that "looks" mean little in these considerations.)

$$\int_0^\infty 4e^{-2x} - 4e^{-3x}\,dx = \lim_{A\to+\infty}\int_0^A 4e^{-2x} - 4e^{-3x}\,dx = \lim_{A\to+\infty} 4\left(\frac{e^{-2x}}{-2} - \frac{e^{-3x}}{-3}\right)\Bigg]_0^A =$$

$\lim_{A\to+\infty} 4\left(\frac{e^{-2A}}{-2} - \frac{e^{-3A}}{-3}\right) - 4\left(\frac{e^0}{-2} - \frac{e^0}{-3}\right) = -4\left(\frac{1}{-2} - \frac{1}{-3}\right) = \frac{2}{3}$, where the very last "simplifying" step isn't necessary, but if you are human, you should have some curiosity about whether the darn area is positive or negative!

(16) 5. Suppose $\mathcal{S}$ is the three-sided region in the first quadrant bounded by the y-axis and the two curves $y = \tan x$ and $y = \sec x$.
a) Sketch that part of the region between $y = 0$ and $y = 5$ on the axes given.
Answer From `maple` (again, not "on the axes given"):



b) Compute the area of the whole region $\mathcal{S}$ (up to ∞) if it is finite.
Answer O.k., let's try it:

$$\int_0^{\pi/2} \sec x - \tan x\,dx = \lim_{B\to\frac{\pi}{2}^-}\int_0^B \sec x - \tan x\,dx =$$

$$\lim_{B\to\frac{\pi}{2}^-} \ln(\sec x + \tan x) - \ln(\sec x)\Big]_0^B = \lim_{B\to\frac{\pi}{2}^-} \ln(1 + \sin x)\Big]_0^B$$

$$= \lim_{B\to\frac{\pi}{2}^-} \ln(1 + \sin B) - \ln 1 = \ln(2)$$

There is something *very* interesting about this computation. Each of the integrals $\int_0^{\pi/2} \sec x\,dx$ and $\int_0^{\pi/2} \tan x\,dx$ does **NOT** converge (you can check this), but their difference does! The sneaky algebra with logs inside the limit is necessary. It is similar to the behavior of the integrals of $\frac{1}{x} + \frac{1}{\sqrt{x}}$ and $\frac{1}{x}$ near 0. Each of those diverges, but their difference *converges*!

(12) 6. a) Suppose m and n are positive integers. Find a reduction formula for

$$\int x^m (\ln x)^n \, dx$$

(Here the object is to reduce n, since if we can push n to 0 we'll just have a polynomial to integrate, which is easy.)

Answer Again, integrate by parts.

$$\underbrace{\int x^m (\ln x)^n \, dx}_{\int u\,dv} = \underbrace{\left(\frac{x^{m+1}}{m+1}\right)\cdot(\ln x)^n}_{uv} - \underbrace{\int n(\ln x)^{n-1}\cdot\left(\frac{1}{x}\right)\left(\frac{x^{m+1}}{m+1}\right)dx}_{\int v\,du} + C$$

$$\left.\begin{array}{r} u = (\ln x)^n \\ dv = x^m \, dx \end{array}\right\}\left\{\begin{array}{l} du = n(\ln x)^{n-1}dx \\ v = \dfrac{x^{m+1}}{m+1} \end{array}\right.$$

I will "clean it up a bit" since this formula will be used in part b):

$$\int x^m (\ln x)^n dx = \left(\frac{x^{m+1}}{m+1}\right)(\ln x)^n - \frac{n}{m+1}\int x^m (\ln x)^{n-1} dx + C$$

b) Use the formula obtained in a) to compute

$$\int x^{20} (\ln x)^2 \, dx$$

Answer Since $n = 2$ we'll need two applications of the formula above.

$$\int x^{20} (\ln x)^2 \, dx = \frac{x^{21}}{21}(\ln x)^2 - \frac{2}{21}\int x^{20}\ln x \, dx = \frac{x^{21}}{21}(\ln x)^2 - \frac{2}{21}\left(\frac{1}{21}x^{21}\ln x - \frac{1}{21}\int x^{20}\,dx\right) =$$
$$\frac{x^{21}}{21}(\ln x)^2 - \frac{2}{21}\left(\frac{1}{21}x^{21}\ln x - \frac{1}{21}\int x^{20}\,dx\right) = \frac{x^{21}}{21}(\ln x)^2 - \frac{2}{21}\left(\frac{1}{21}x^{21}\ln x - \frac{1}{21}\cdot\frac{x^{21}}{21}\right) + C$$

I doubt whether I would "simplify" anything here except under duress.

(14) 7. This problem analyzes the computation needed to estimate the definite integral

$$\int_0^1 \frac{1}{9}\sin\left(x^3\right) dx$$

a) Find n (the number of subdivisions) so that the Trapezoidal Rule estimate will be within 10^{-6} of the true value of the definite integral. (You may use the error bound $\frac{K(b-a)^3}{12n^2}$ where K is an overestimate of the magnitude of the second derivative.)

DO NOT COMPUTE THE TRAPEZOIDAL RULE ESTIMATE.

Answer If $f(x) = \frac{1}{9}\sin\left(x^3\right)$ then $f'(x) = \frac{1}{3}\cos\left(x^3\right)x^2$ and $f''(x) = -\sin\left(x^3\right)x^4 + \frac{2}{3}\cos\left(x^3\right)x$ and (we'll need it eventually!)

$f^{(3)}(x) = -3\cos\left(x^3\right)x^6 - 4\sin\left(x^3\right)x^3 - 2\cos\left(x^3\right)x^3 + \frac{2}{3}\cos\left(x^3\right)$ so, simplifying a bit:

$f^{(3)}(x) = -3\cos\left(x^3\right)x^6 - 6\sin\left(x^3\right)x^3 + \frac{2}{3}\cos\left(x^3\right)$ and even more:

$f^{(4)}(x) = 9\sin\left(x^3\right)x^8 - 18\cos\left(x^3\right)x^5 - 18\sin\left(x^3\right)x^5 - 18\sin\left(x^3\right)x^2 - 2\sin\left(x^3\right)x^2$ so, again simplifying,

$f^{(4)}(x) = 9\sin\left(x^3\right)x^8 - 18\cos\left(x^3\right)x^5 - 18\sin\left(x^3\right)x^5 - 20\sin\left(x^3\right)x^2$.

I admit that I checked these derivatives with `maple`, but I did them myself first.

Comment The "point" of the $\frac{1}{9}$ in the original function was to reduce the integers involved in the fourth derivative (otherwise they would be 9 times larger). I think this reduction was worth it but I am not sure. In any case, this problem is certainly not the most difficult on the exam! Let's continue:

The functions involved are very easy to estimate on $[0,1]$. I will assume no cancellation, and just realize that $x^{\text{positive power}}$ and sin(anything) and cos(anything) are all bounded in absolute value by 1 on $[0,1]$. Note that to successfully complete this problem, I *must* give secure estimates of the various derivatives. The estimates obtained on currently available graphing calculators (using numerical approximations for derivatives of functions) are *not* secure, because they are just approximations. If I want to know that a certain n is guaranteed to work so that a certain estimate is definitely within the desired tolerance, then I must do an analysis similar to what is done here.
Therefore $|f''(x)| \leq 1 + \frac{2}{3} \leq 2$ for x in $[0,1]$, so I'll take $K = 2$. I don't need the best possible K or the smallest valid n – you are not asked for that!
Then $\dfrac{K(b-a)^3}{12n^2}$ becomes $\dfrac{2}{12n^2}$. What n will force this to be certainly less than 10^{-6}?
$\dfrac{1}{6n^2} \leq 10^{-6}$ becomes $\sqrt{\dfrac{10^6}{6}} \leq n$ so any n larger than about 410 will do. At last a use for a calculator!

b) Find n (the number of subdivisions) so that the Simpson's Rule estimate will be within 10^{-6} of the true value of the definite integral. (You may use the error bound $\dfrac{J(b-a)^5}{180n^4}$ where J is an overestimate of the magnitude of the fourth derivative.)

DO NOT COMPUTE THE SIMPSON'S RULE ESTIMATE.

Answer Finding J will almost bring us to the end. A look at $f^{(4)}(x)$ shows that $9+18+18+20 = 65$ gives a simple and quick overestimate of $|f^{(4)}(x)|$ for x in $[0,1]$. So we can take $J = 65$.
Then $\dfrac{J(b-a)^5}{180n^4}$ becomes $\dfrac{65}{180n^4}$. What n will force this to be certainly less than 10^{-6}?
$\dfrac{65}{180n^4} \leq 10^{-6}$ becomes $\left(\dfrac{65 \cdot 10^6}{180}\right)^{\frac{1}{4}} \leq n$ so any n larger than about 26 (n should be even for Simpson's Rule) will be adequate.

Comment I don't know if the difference between 26 and 410 is enough compensation for the effort of computing two more derivatives. Perhaps if function values are difficult or expensive to obtain, it is. Or if computational time is inadequate and you need to do similar integrals 10^{10} times each day, then ...