Some answers to the quiz for 152:5-7 and 9-11

1. The integral \( \int_{0}^{2} \frac{1}{\sqrt{2-x}} \, dx \) is an improper integral.

You must analyze this improper integral using the following sequence of steps:

**First** Write this integral as a limit of proper definite integrals with a varying parameter. **Answer** The interval is finite, but the function’s domain does not include 2, which is an endpoint of the interval. Therefore we should make the definite integral range from 0 to a number less than 2. Here is an acceptable answer:

\[
\int_{0}^{2} \frac{1}{\sqrt{2-x}} \, dx = \lim_{A \to 2^-} \int_{0}^{A} \frac{1}{\sqrt{2-x}} \, dx
\]

**Second** Evaluate the definite integral with a parameter which appears inside the limit (neither the word “limit” nor the term lim\(A \to 2^-\) should appear in this stage). Your answer should include one or more expressions with the parameter.

\[
\int_{0}^{A} \frac{1}{\sqrt{2-x}} \, dx = \int_{0}^{A} (2-x)^{-\frac{1}{2}} \, dx = -2 (2-x)^{\frac{1}{2}} \bigg|_{0}^{A} = -2\sqrt{2-A} - (-2\sqrt{2-0}) = -2\sqrt{2-A} + 2\sqrt{2}
\]

**Third** Use the previously computed answer and the limit expression you got in the first part of this problem to decide if the improper integral converges, and, if it does, find the value of the integral.

\[
\lim_{A \to 2^-} \int_{0}^{A} \frac{1}{\sqrt{2-x}} \, dx = \lim_{A \to 2^-} -2\sqrt{2-A} + 2\sqrt{2} = 2\sqrt{0} + 2\sqrt{2} = 2\sqrt{2}
\]
2. The horizontal and vertical axes on the graph below have different scales. The graph is a direction field for the differential equation

\[ y' = -\frac{1}{30}(y - 1)^2(2 + y). \]

a) Find the equilibrium solutions (where \( y \) doesn’t change) for this differential equation.

**Answer** Here \( y' = 0 \) always so \( y \) must be constant. The solutions are \( y(x) = 1 \) (which I call \( E_1 \)) and \( y(x) = -2 \) (which I call \( E_2 \)).

b) Sketch solution curves on the axis above through these points. Find the indicated limits:

- \((0, 0)\). Label this curve \( A \). On curve \( A \), \( \lim_{x \to \infty} y(x) = -2 \).
- \((0, 2)\). Label this curve \( B \). On curve \( B \), \( \lim_{x \to \infty} y(x) = 1 \).
- \((0, -3)\). Label this curve \( C \). On curve \( C \), \( \lim_{x \to \infty} y(x) = -2 \).
- \((1, 0)\). Label this curve \( D \). On curve \( D \), \( \lim_{x \to \infty} y(x) = -2 \).

**Comment** Both of the pictures shown were drawn by “machine”. I don’t think it would be very easy to get formulas for the solution curves, so the curves were drawn using a sophisticated numerical approximation program. One surprise was how close the curves \( A \) and \( D \) appeared. In retrospect, the graphs seem to almost touch because of the difference in the horizontal and vertical scales.

c) One of the equilibrium solutions is a *stable* equilibrium. Which one?

**Answer** \( E_2 \) or \( y = -2 \).