Answers, Review Final Exam — Math 151 — Fall Semester 2007

1(a) $e^{e-1}/2$. $\lim\limits_{x\to e}\dfrac{e^x-x^e}{(x-e)^2}\overset{L'Hosp.}{=}\lim\limits_{x\to e}\dfrac{e^x-ex^{e-1}}{2(x-e)}\overset{L'Hosp.}{=}\lim\limits_{x\to e}\dfrac{e^x-e(e-1)x^{e-2}}{2}=$ $(e^e-e(e-1)e^{e-2})/2=(e^e-(e-1)e^{e-1})/2=e^{e-1}/2$.

1(b) 1. Let $y=(1+x)^{1/x}$. Taking ln( ), $\ln y=\ln((1+x)^{1/x})=\dfrac{\ln(1+x)}{x}$. Then, $\lim\limits_{x\to\infty}\ln y=\lim\limits_{x\to\infty}\dfrac{\ln(1+x)}{x}\overset{L'Hosp.}{=}\lim\limits_{x\to\infty}\dfrac{1/(1+x)}{1}=0$. Thus, as $x\to\infty$, $\ln y\to 0$. Thus, $y=e^{\ln y}\to e^0=1$.

1(c) 1. $\lim\limits_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)=\lim\limits_{x\to\infty}\left((\sqrt{x^2+x}-\sqrt{x^2-x})\cdot\dfrac{(\sqrt{x^2+x}+\sqrt{x^2-x})}{(\sqrt{x^2+x}+\sqrt{x^2-x})}\right)=$ $\lim\limits_{x\to\infty}\left(\dfrac{(x^2+x)-(x^2-x)}{\sqrt{x^2+x}+\sqrt{x^2-x}}\right)=\lim\limits_{x\to\infty}\dfrac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}}=\lim\limits_{x\to\infty}\dfrac{2x}{\sqrt{x^2(1+1/x)}+\sqrt{x^2(1-1/x)}}=$ $\lim\limits_{x\to\infty}\dfrac{2x}{x(\sqrt{1+1/x}+\sqrt{1-1/x})}=\lim\limits_{x\to\infty}\dfrac{2}{\sqrt{1+1/x}+\sqrt{1-1/x}}=\dfrac{2}{1+1}=1$, as $\lim\limits_{x\to\infty}1/x=0$.

2(a) $\dfrac{d}{dx}\left(\dfrac{x^2+x+1}{x^3+x}\right)=\dfrac{(x^3+x)(2x+1)-(x^2+x+1)(3x^2+1)}{(x^3+x)^2}=$ $\dfrac{(2x^4+2x^2+x^3+x)-(3x^4+3x^3+3x^2+x^2+x+1)}{(x^3+x)^2}=\dfrac{-x^4-2x^3-2x^2-1}{(x^3+x)^2}$.

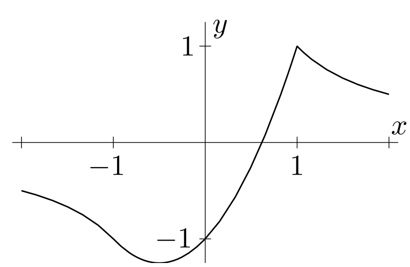
2(b) $\dfrac{d}{dx}(x\tan(e^{-x}))=1\cdot\tan(e^{-x})+x\dfrac{d}{dx}(\tan(e^{-x}))=\tan(e^{-x})+x\sec^2(e^{-x})\dfrac{d}{dx}(e^{-x})=$ $\tan(e^{-x})+x\sec^2(e^{-x})(-e^{-x})=\tan(e^{-x})-xe^{-x}\sec^2(e^{-x})$.

2(c) Let $y=(1+x+x^2)^x$. Taking ln( ), $\ln y=\ln(1+x+x^2)^x=x\ln(1+x+x^2)$. Thus, differentiating with respect to x, it follows that $\dfrac{1}{y}\cdot\dfrac{dy}{dx}=\ln(1+x+x^2)+x\cdot\dfrac{d}{dx}(\ln(1+x+x^2))=$ $\ln(1+x+x^2)+x\cdot\dfrac{1+2x}{1+x+x^2}=\ln(1+x+x^2)+\dfrac{x+2x^2}{1+x+x^2}$.

Thus, $\dfrac{dy}{dx}=\left(\ln(1+x+x^2)+\dfrac{x+2x^2}{1+x+x^2}\right)(1+x+x^2)^x$.

2(d) $\sqrt{1+x^4}$, by the Fundamental Theorem of Calculus.

3(a) Suppose that f is continuous everywhere. Then, $\lim\limits_{x\to 1}f(x)$ exists and equals $f(1)$. In particular, the left and right hand limits at $x=1$ are equal. Thus, $\lim\limits_{x\to 1^+}f(x)=\lim\limits_{x\to 1^-}f(x)$. Near $x=1$, to the right of 1, $f(x)=1/x$, and to the left of 1, $f(x)=x^2+Ax+B$. It follows that: $\lim\limits_{x\to 1^+}1/x=\lim\limits_{x\to 1^-}(x^2+Ax+B)$, or $1=1+A+B$, or $A+B=0$.



Similarly, $\lim\limits_{x\to(-1)^+}(x^2+Ax+B)=\lim\limits_{x\to(-1)^-}1/x$. Thus, $1-A+B=-1$, or $-A+B=-2$. Since $B=-A$, $-2A=-2$, or $A=1$. Also, $B=-1$.

3(b) The graph appears above and to the right. The function fails to be differentiable at $x=1$. If $-1\le x\le 1$, $f'(x)=2x+1$. If $x<1$, $f'(x)=-1/x^2$. At $x=-1$, both are -1, and f is differentiable at $x=-1$. At all other points, f is differentiable.

4. Here, $f'(x)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=$ $\lim\limits_{h\to 0}\dfrac{1}{h}\left(\dfrac{1}{(2x+2h+1)^2}-\dfrac{1}{(2x+1)^2}\right)=\lim\limits_{h\to 0}\dfrac{1}{h}\left(\dfrac{(2x+1)^2-(2x+2h+1)^2}{(2x+1)^2(2x+2h+1)^2}\right)=$

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{(4x^2+4x+1)-(4x^2+4h^2+1+8xh+4x+4h)}{(2x+1)^2(2x+2h+1)^2}\right)=\lim_{h\to 0}\frac{1}{h}\left(\frac{-4h^2-8xh-4h}{(2x+1)^2(2x+2h+1)^2}\right)$$
$$=\lim_{h\to 0}\frac{-4h-8x-4}{(2x+1)^2(2x+2h+1)^2}=\frac{-8x-4}{(2x+1)^4}=\frac{-4(2x+1)}{(2x+1)^4}=\frac{-4}{(2x+1)^3}.$$

5(a) Differentiating with respect to x: $3x^2+6xy+3x^2\dfrac{dy}{dx}+3y^2\dfrac{dy}{dx}=0$, or

$(3x^2+3y^2)\dfrac{dy}{dx}=-3x^2-6xy$. Thus, $\dfrac{dy}{dx}=\dfrac{-3x^2-6xy}{3x^2+3y^2}=-\dfrac{x^2+2xy}{x^2+y^2}$.

5(b) A point on the tangent line is $(1,1)$. To find the slope of the tangent line we evaluate the derivative at $(1,1)$. Letting $x=1$ and $y=1$ in the formula for dy/dx, then $dy/dx=(-1-2)/2=-3/2$. Thus, the equation of the tangent line is $y-1=(-3/2)(x-1)$, or $y=(-3x+5)/2$.

5(c) The tangent line is horizontal if $dy/dx=0$, i.e., $x^2+2xy=0$, or $x(x+2y)=0$. Thus, either $x=0$ or $x=-2y$.

If $x=0$, then from $x^3+3x^2y+y^3=5$, $y^3=5$, or $y=\sqrt[3]{5}$. Thus, $(x,y)=(0,\sqrt[3]{5})$.

If $x=-2y$, then from $x^3+3x^2y+y^3=5$, $(-2y)^3+3(-2y)^2y+y^3=5$, or $(-8+12+1)y^3=5$, or $5y^3=5$, or $y=1$. Then $x=-2$.

Thus, either $(x,y)=(0,\sqrt[3]{5})$ or $(x,y)=(-2,1)$.

6. Here, the volume, V, and the surface area, S, of a sphere of radius r are given by the formulas: $V=\dfrac{4}{3}\pi r^3$, $S=4\pi r^2$. In addition, by hypothesis, $\dfrac{dV}{dt}=5$ and $\dfrac{dS}{dt}=3$. Differentiating the formulas with respect to t, $\dfrac{dV}{dt}=4\pi r^2\dfrac{dr}{dt}=4\pi r^2r'$, and $\dfrac{dS}{dt}=8\pi r\dfrac{dr}{dt}=8\pi rr'$. Thus, $5=4\pi r^2r'$ and $3=8\pi rr'$. Dividing, $5/3=(4\pi r^2r')/(8\pi rr')$, or $5/3=r/2$. Thus, $r=10/3$ (inches).

7(a) $y=\ell_1(x)=-3+(x+2)/2=x/2-2$, $y=\ell_2(x)=-3+x+2=x-1$.

7(b) Suppose that $-2<x$. By the Mean Value Theorem, $f(x)-f(-2)=f'(c)(x+2)$, with $-2<c<x$. Since $-2<c$, $1/2\le f'(c)\le 1$. Since $-2<x$, $0<x+2$. Multiplying by $x+2$, $(x+2)/2\le f'(c)(x+2)\le(x+2)$. Thus, $(x+2)/2\le f(x)-f(-2)\le(x+2)$, or $f(-2)+(x+2)/2\le f(x)\le f(-2)+(x+2)$. Thus, replacing $f(-2)$ by -3, it follows that $-3+(x+2)/2=\ell_1(x)\le f(x)\le\ell_2(x)=-3+(x+2)$.

8. At $x=-5$ and $x=3$, $f'(x)$ changes sign from positive to negative, and thus a local maximum occurs at $x=-5$ and $x=3$.

At $x=0$ and $x=6$, $f'(x)$ changes sign from negative to positive, and thus a local minimum occurs at $x=0$ and $x=6$.

At $x=-3$ and $x=1$, $f'(x)$ does not change sign, and neither occurs.

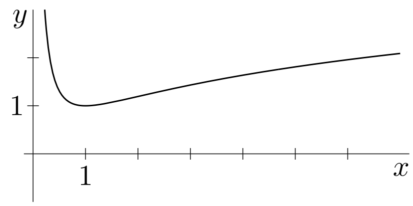
9(a) Here, $y=f(x)=\dfrac{1}{x}+\ln x$. The domain is $(0,\infty)$, since $\ln x$ is only defined if $x>0$.

As $x\to 0^+$, $y\to\infty-\infty$, and to evaluate the limit, we collect the terms together. Then, $\displaystyle\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}\left(\frac{1+x\ln x}{x}\right)$. Observe that $\displaystyle\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\left(\frac{\ln x}{1/x}\right)$. As this is $\dfrac{-\infty}{\infty}$, we use L'Hospital's rule to get: $\displaystyle\lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)=\lim_{x\to 0^+}-x=0$. Thus, as $x\to 0^+$, $x\ln x\to 0$. Then, as $x\to 0^+$, $1+x\ln x\to 1+0=1$. It follows that $\displaystyle\lim_{x\to 0^+}\left(\frac{1+x\ln x}{x}\right)=+\infty$. Thus, the graph has a vertical asymptote at $x=0$.

Since $\displaystyle\lim_{x\to\infty}\left(\frac{1}{x}+\ln x\right)=\infty$, the graph has no horizontal asymptote.

Differentiating, $\dfrac{dy}{dx} = -\dfrac{1}{x^2} + \dfrac{1}{x} = \dfrac{-1+x}{x^2}$. Thus, if $dy/dx = 0$, $x = 1$, and the only critical point is $x = 1$. Also, $\dfrac{d^2y}{dx^2} = \dfrac{2}{x^3} - \dfrac{1}{x^2} = \dfrac{2-x}{x^3}$. Thus, if $d^2y/dx^2 = 0$, $x = 2$, and the only inflection point is $x = 2$.

Since $\lim\limits_{x\to 0^+} f(x) = \infty$ and $\lim\limits_{x\to\infty} f(x) = \infty$, the function has a minimum at $x = 1$. Thus, as $f(1) = 1$, the range of the function is $[1, \infty)$. The graph of the function appears to the right.

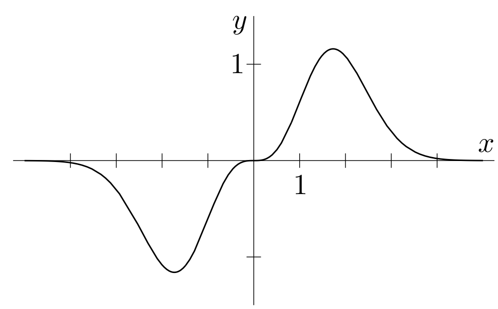


9(b) For this function, the domain is all real numbers and there are no vertical asymptotes. Also, $dy/dx = 3x^2e^{-x^2/2} + x^3(-x)e^{-x^2/2} = (3x^2 - x^4)e^{-x^2/2}$. Thus, if $dy/dx = 0$, then $x^4 - 3x^2 = x^2(x^2-3) = 0$. It follows that $x = 0$ or $x = \pm\sqrt{3}$. Thus, the critical points of the function are $x = 0$ and $x = \pm\sqrt{3}$.

Similarly, $d^2y/dx^2 = (6x - 4x^3)e^{-x^2/2} + (3x^2 - x^4)(-x)e^{-x^2/2} = (x^5 - 7x^3 + 6x)e^{-x^2/2}$. Thus to be an inflection point, x must satisfy the equation $x^5 - 7x^3 + 6x = 0$, or $x(x^2-1)(x^2-6) = 0$. Thus, the inflection points are $x = 0$, $x = \pm 1$, and $x = \pm\sqrt{6}$.

Since $\lim\limits_{x\to\infty} x^3e^{-x^2/2} = \lim\limits_{x\to\infty} \dfrac{x^3}{e^{x^2/2}} = \lim\limits_{x\to\infty} \dfrac{3x^2}{-xe^{x^2/2}} = \lim\limits_{x\to\infty} \dfrac{-3x}{e^{x^2/2}} = \lim\limits_{x\to\infty} \dfrac{-3}{-xe^{x^2/2}} = 0$, the horizontal asymptote of the graph is the line $y = 0$.

From the graph, the maximum value of the function occurs at $x = \sqrt{3}$ and is $M = 3^{3/2}e^{-3/2} = (3/e)^{3/2}$. Likewise, the minimum is $-3^{3/2}e^{-3/2} = -(3/e)^{3/2}$. Thus, the range is the interval $[-(3/e)^{3/2}, (3/e)^{3/2}] = [-1.159418, 1.159418]$.



10(a) Since the linear approximation at $x = a$ of f is $L(x) = f(a) + f'(a)(x-a)$, with $a = 1$, $f(1) = 3$, $f'(1) = -2$, $L(x) = 3 - 2(x-1)$.

10(b) Setting $x = 1.05$, $L(1.05) = 3 - 2(1.05 - 1) = 2.9$

10(c) Here $g(x) = x^2 f(x)$. Then $g(1) = 1^2 f(1) = 3$. Also $g'(x) = 2x\,f(x) + x^2 f'(x)$. Setting $x = 1$, $g'(1) = 2f(1) + f'(1) = 2\cdot 3 - 2 = 4$. Thus, denoting the linear approximation of $g(x)$ at $a = 1$ by $M(x)$, then $M(x) = 3 + 4(x-1)$.

11(a) $\displaystyle\int \frac{x^2+x+1}{\sqrt{x}}\,dx = \int (x^{3/2} + x^{1/2} + x^{-1/2})\,dx = (2/5)x^{5/2} + (2/3)x^{3/2} + 2x^{1/2} + C.$

11(b) With $u = x + 1$, $du = dx$. Thus, $\displaystyle\int \sqrt{x+1}\,dx = \int \sqrt{u}\,du = (2/3)u^{3/2} + C =$ $(2/3)(x+1)^{3/2} + C$. Thus, $\displaystyle\int_0^1 \sqrt{x+1}\,dx = (2/3)(x+1)^{3/2}\Big|_0^1 = (2/3)(2^{3/2} - 1).$

11(c) Take $u = x^3$. Then, $du = 3x^2dx$, or $x^2dx = (1/3)du$.

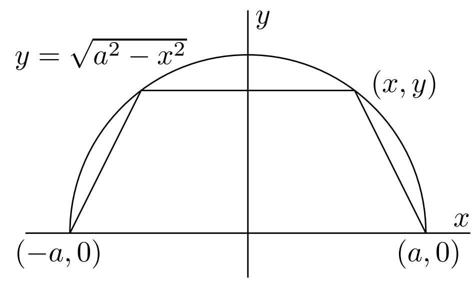
Then, $\displaystyle\int x^2\sin(x^3)\,dx = \int (1/3)\sin\,u\,du = -(1/3)\cos\,u + C = -(1/3)\cos(x^3) + C.$

11(d) Let $u = \ln\,x$. Then, $\dfrac{du}{dx} = \dfrac{1}{x}$ and $du = \dfrac{dx}{x}$. It follows that

$$\int \frac{\cos(\ln\,x)}{x}\,dx = \int \cos(u)\,du = \sin(u) + C = \sin(\ln\,x) + C.$$

Thus, $\displaystyle\int_1^e \frac{\cos(\ln\,x)}{x}\,dx = \sin(\ln\,x)\Big|_1^e = \sin(\ln\,e) - \sin(\ln\,1) = \sin(1) - \sin(0) = \sin(1).$

12(a) Here, $a(t) = \dfrac{d^2x}{dt^2} = -24t^2 + 96t$. Thus, $v(t) = \dfrac{dx}{dt} = \displaystyle\int (-24t^2 + 96t)\,dt = -8t^3 + 48t^2 + C$, where C is a constant. Since $v(0) = 0$, $0 = -8{\cdot}0^3 + 48{\cdot}0^2 + C$, and $C = 0$.
Thus, $v(t) = \dfrac{dx}{dt} = -8t^3 + 48t^2$, and so, $x(t) = \displaystyle\int (-8t^3 + 48t^2)\,dt = -2t^4 + 16t^3 + D$, with D a constant. Since $x(0) = 0$, $0 = 0 + 0 + D$, or $D = 0$. It follows that $x(t) = -2t^4 + 16t^3$.
12(b) The particle goes down the x-axis until $dx/dt = 0$. Since $dx/dt = -8t^3 + 48t^2$, $dx/dt = 0$ if $t = 0$ or $t = 6$. At $t = 0$, $x = 0$, and at $t = 6$, $x = 864$. Thus, $x = 864$ is as far as the particle goes down the positive x-axis.



13. Let (x, y) be the upper right hand corner of the trapezoid. Then, the top side of the trapezoid has width $2x$, the bottom side has width $2a$, and the height is y. Thus, if A is the area $A = (2x+2a)y/2 = (x+a)y$. Since (x, y) is a point on the semi-circle, $y = \sqrt{a^2 - x^2}$ and $A = (x + a)\sqrt{a^2 - x^2}$

Then, $\dfrac{dA}{dx} = \sqrt{a^2 - x^2} + (x + a)\left(\dfrac{-x}{\sqrt{a^2 - x^2}}\right) = \dfrac{(a^2 - x^2) - x(x + a)}{\sqrt{a^2 - x^2}} = \dfrac{-2x^2 - ax + a^2}{\sqrt{a^2 - x^2}} = \dfrac{-2(x^2 + (a/2)x - a^2/2)}{\sqrt{a^2 - x^2}} = \dfrac{-2(x + a)(x - a/2)}{\sqrt{a^2 - x^2}}$. Thus, the critical points of A are $x = -a$ and $x = a/2$. Since the problem only makes sense for $0 \le x \le a$, the relevant critical point is $x = a/2$.

At the endpoint $x = 0$, $A(0) = a^2$. At the critical point $x = a/2$, $A(a/2) = 3\sqrt{3}a^2/4$. Finally, at the endpoint $x = a$, $A(a) = 0$. Thus, the maximum value is $A = 3\sqrt{3}a^2/4$, and it occurs at $x = a/2$.
14. The area of a semi-circle of radius 1 is $\pi/2$. The trapezoidal regions below $[2, 5]$ and $[7, 10]$ each have total area 2.
(i) $\pi/2$ *(ii)* $\pi/2 - 2$ *(iii)* $\pi - 2$ *(iv)* $\pi - 4$ *(v)* $\pi/2 - 3$
15. Since $[1, 3]$ is divided into four subintervals of equal length, the width of each subinterval is $\Delta x = (3 - 1)/4 = 1/2$. Writing 1 as $2/2$ and 3 as $6/2$, the subintervals are:
$[2/2, 3/2]$, $[3/2, 4/2]$, $[4/2, 5/2]$, $[5/2, 6/2]$. Here, also $f(x) = \dfrac{1}{x}$ and $\Delta x = 1/2$.
15(a) The left endpoints of the subintervals are: $2/2, 3/2, 4/2, 5/2$. Thus, the Riemann sum is:
$(1/2)(f(2/2) + f(3/2) + f(4/2) + f(5/2)) = \dfrac{1}{2}\left(\dfrac{2}{2} + \dfrac{2}{3} + \dfrac{2}{4} + \dfrac{2}{5}\right) = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} = \dfrac{77}{60}$.
15(b) The right endpoints of the subintervals are: $3/2, 4/2, 5/2, 6/2$. Thus, the Riemann sum is:
$(1/2)(f(3/2) + f(4/2) + f(5/2) + f(6/2)) = \dfrac{1}{2}\left(\dfrac{2}{3} + \dfrac{2}{4} + \dfrac{2}{5} + \dfrac{2}{6}\right) = \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \dfrac{1}{6} = \dfrac{57}{60}$.
15(c) The midpoints of the subintervals are: $5/4, 7/4, 9/4, 11/4$. Thus, the Riemann sum is:
$(1/2)(f(5/4)+f(7/4)+f(9/4)+f(11/4)) = \dfrac{1}{2}\left(\dfrac{4}{5} + \dfrac{4}{7} + \dfrac{4}{9} + \dfrac{4}{11}\right) = 2\left(\dfrac{1}{5} + \dfrac{1}{7} + \dfrac{1}{9} + \dfrac{1}{11}\right) = \dfrac{3776}{3465}$.

16. The parabola $y = -x^2 + 5x - 3$ crosses the line $y = x$ at those points x where $-x^2 + 5x - 3 = x$, or $x^2 - 4x + 3 = 0$. Thus, factoring, $(x - 1)(x - 3) = 0$, or $x = 1$ and $x = 3$. Thus, the area is:

$$\int_1^3 ((-x^2 + 5x - 3) - (x))\,dx = \int_1^3 (-x^2 + 4x - 3)\,dx =$$

$$(-x^3/3 + 2x^2 - 3x)\Big|_1^3 = 4/3.$$