640:135:21, 22, & 23  {\textbf{Answers to version A of the first exam}}  2/22/2005

(10)  1. Suppose \( f(x) = \frac{1}{x + 2} \). Use the definition of derivative to find \( f'(x) \). \textbf{Answer}  Since \( f(x) = \frac{1}{x + 2} \), \( f(x+h) = \frac{1}{x + h + 2} \) and \( f(x+h) - f(x) = \frac{1}{x + h + 2} - \frac{1}{x + 2} = \frac{(x + 2) - (x + h + 2)}{(x + h + 2)(x + 2)} = \frac{-h}{(x + h + 2)(x + 2)} \). Therefore \( \frac{f(x+h) - f(x)}{h} = \frac{-1}{(x + h + 2)(x + 2)} \) and the limit of this as \( h \to 0 \) is \( \frac{-1}{(x + 2)^2} \), which is \( f'(x) \).

(12)  2. a) Find the equation of a line tangent to \( y = \frac{4x}{2+x^2} \) when \( x = 1 \).
\textbf{Answer}  When \( x = 1 \), \( y = \frac{4 \cdot 1}{2 + 1^2} = \frac{4}{3} \). And \( y' = \frac{4(2 + x^2) - 2x(4x)}{(2 + x^2)^2} \). When \( x = 1 \), \( y' = \frac{4(2 + 1^2) - 2(4)}{(2 + 1^2)^2} = \frac{4}{9} \). Therefore the tangent line has equation \( y - \frac{4}{3} = \frac{4}{9}(x - 1) \).

b) Sketch the graph of \( y = \frac{4x}{2+x^2} \) and the tangent line found in a) on the axes below. \textbf{Answer}  A picture is to the right.

(14)  3. Suppose that the function \( f(x) \) is described by \( f(x) = \begin{cases} 3 - x^2 & \text{if } x < 0 \\ Ax + B & \text{if } 0 \leq x \leq 1 \\ 2^x & \text{if } 1 < x \end{cases} \).

a) Find \( A \) and \( B \) so that \( f(x) \) is continuous for all numbers. Briefly explain your answer.
\textbf{Answer}  \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \) and continuity implies this should equal \( \lim_{x \to 0} f(x) = \lim_{x \to 1^-} Ax + B = B \). So \( B \) must be 3. Also \( \lim_{x \to 0^+} f(x) = \lim_{x \to 1^-} Ax + B = A + B \), and again continuity makes this the same as \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} 2^x = 2 \). Therefore \( A + B = 2 \) and since \( B = 3 \), \( A \) must be \(-1\).

b) Sketch \( y = f(x) \) on the axes given for the values of \( A \) and \( B \) found in a) when \( x \) is in the interval \([-2, 2]\).
\textbf{Answer}  On the interval \([-2, 0]\), the graph of \( y = 3 - x^2 \) is drawn. On \([0, 1]\), the graph of \( y = -x + 3 \) is drawn. On \([1, 2]\), the graph of \( y = 2^x \) is drawn. These curves all meet and the graph shows a continuous function.

(20)  4. Evaluate the indicated limits exactly. Give evidence to support your answers.

a) \( \lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1} \)
\textbf{Answer}  Since \( x^2 + 2x - 3 = (x + 3)(x - 1), \frac{x^2 + 2x - 3}{x - 1} \) is \( x + 3 \) away from 1. Therefore we need \( \lim_{x \to 1} x + 3 \) which is 4.

b) \( \lim_{x \to 2^+} \frac{|x - 1| - 1}{|x - 2|} \)
\textbf{Answer}  As \( x \to 2^+ \), we know \( x > 2 \). Therefore \( x - 2 > 0 \) and \( |x - 2| = x - 2 \). Also certainly \( |x - 1| = x - 1 \) when \( x > 2 \) so that \( \frac{|x - 1| - 1}{|x - 2|} = \frac{x - 1 - 1}{x - 2} = 1 \) for \( x > 2 \). The value of the limit is 1.

c) \( \lim_{x \to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3} \)
\textbf{Answer}  Multiply \( \frac{\sqrt{x} - \sqrt{3}}{x - 3} \) by \( \frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}} \) and get \( \frac{x - 3}{(x - 3) \cdot (\sqrt{x} + \sqrt{3})} \) which away from 3 is \( \frac{1}{\sqrt{x} + \sqrt{3}} \). This has limit \( \frac{1}{2\sqrt{3}} \) as \( x \to 3 \).

d) \( \lim_{x \to 4} \frac{3x - 2}{\cos(\pi x)} \)
\textbf{Answer}  This function is continuous at 4, so its limit is its value at 4: \( \frac{3 \cdot 4 - 2}{\cos(4 \pi)} = \frac{10}{\cos(4 \pi)} = 10 \).

\textbf{OVER}
5. Suppose that \( f(x) = x^5 + 3\cos(K(x^2)) \) where \( K \) is an unknown constant.
a) Briefly explain why \( f(2) \) must be positive.
Answer We know that values of cosine must be between \(-1\) and \(1\), and therefore \( 3\cos(K(2^2)) \) must be between \(-3\) and \(3\). Since \(2^5 = 32\), the value of \( f(2) \) is between \(29\) and \(35\). So \( f(2) \) must be positive.
b) Briefly explain why \( f(-2) \) must be negative.
Answer We know that values of cosine must be between \(-1\) and \(1\), and therefore \( 3\cos(K(-2)^2) \) must be between \(-3\) and \(3\). Since \((-2)^5 = -32\), the value of \( f(2) \) is between \(-35\) and \(-29\). So \( f(2) \) must be positive.
c) Briefly explain why the equation \( f(x) = 0 \) must have a solution, and specify an interval on the \(x\)-axis in which a solution can be found.
Answer \( f(x) \) is a continuous function and \( f(-2) \) is negative and \( f(2) \) is positive. Therefore, by the Intermediate Value Theorem, \( f(x) = 0 \) must have at least one root inside the interval \([-2, 2]\).

6. What is the domain of \( f(x) = \frac{\ln x + \sqrt{4-x}}{\sin x} \)? Explain your answer algebraically.
Answer Certainly \( x > 0 \) so that \( \ln x \) will be defined. Also \( \sqrt{4-x} \) will be defined only when \( 4-x \geq 0 \) which is when \( 4 \geq x \). Therefore the top of the fraction defining \( f(x) \) restricts the domain to \( 0 < x \leq 4 \). The bottom, \( \sin x \), should not be zero. The only \( x \) in \( 0 < x \leq 4 \) for which \( \sin x = 0 \) is \( x = \pi \). Therefore the domain is the two intervals \( 0 < x < \pi \) and \( \pi < x \leq 4 \).

7. In this problem the function \( f(x) \) has domain all \( x \)'s between \(-4 \) and \(4: -4 < x < 4 \). A graph of \( y = f(x) \) is displayed below. Answer the following questions as well as you can based on the information in the graph.
a) What is the range (the collection of values) of \( f(x) \)?
Answer Horizontal lines with \( y = a \) a constant intersect the graph for \( x \)'s in the interval \(-2 < x \leq 3 \), which is the range.
b) For which \( x \) is \( f(x) \) not continuous?
Answer Only at \( x = -1 \) where \( f(x) \) has a jump.
c) For which \( x \) is \( f(x) = 0 \)?
Answer \( x = -3 \)
d) For which \( x \) is \( f(x) > 0 \)?
Answer \(-3 < x < 4 \)
e) For which \( x \) is \( f(x) \) not differentiable?
Answer \( x = -1 \) (where the function is not continuous and therefore can’t be differentiable) and \( x = 2 \), where the graph has a corner.
f) For which \( x \) is \( f'(x) = 0 \)?
Answer \( x = -2 \)
g) For which \( x \) is \( f'(x) > 0 \)?
Answer The two intervals \(-4 < x < -2 \) and \(2 < x < 4 \)

8. a) If \( f(x) = \frac{1 - e^x}{x^2 + 1} \), what is \( f'(x) \)?
Answer \( f'(x) = \frac{(-e^x)(x^2 + 1) - (2x)(1 - e^x)}{(x^2 + 1)^2} \)
b) If \( f(x) = (2x + 3\cos x)(x^4 - x^2) \), what is \( f'(x) \)?
Answer \( f'(x) = (2 - 3\sin x)(x^4 - x^2) + (2x + 3\cos x)(4x^3 - 2x) \)
c) Suppose that \( g(x) \) is a differentiable function and that \( g(1) = 2, g'(1) = -3, \) and \( g''(1) = 4 \). What is the value of the second derivative of \( f(x) = x^3g(x) \) when \( x = 1? \)
Answer Since \( f(x) = x^3g(x) \), I know that \( f'(x) = 3x^2g(x) + x^3g'(x) \) and then \( f''(x) = 6xg(x) + 3x^2g'(x) + 3x^2g''(x) + x^3g'''(x) \). The Product Rule has been used three times, once for computing \( f'(x) \) and twice for computing \( f''(x) \). Thus \( f''(1) = 6 \cdot 1 \cdot 2 + 3 \cdot 1^2 \cdot (-3) + 3 \cdot 1^2 \cdot 4 = -2 \).