Math 504: Complex Variables (Spring, 2000)

B1 Prove that the only compactly supported real analytic function on \(\mathbb{R}\) is the zero function. Prove that the only compactly supported real analytic function on \(\mathbb{R}^2\) is the zero function.

B2 Suppose that a power series \(\sum_{(j,k)=(0,0)}^{\infty} \nu_{j,k}x^jy^k\) converges in some neighborhood of \((0,0) \in \mathbb{R}^2\), where the \(\nu_{j,k}\) are complex constants. What conditions on these constants are needed to insure that the sum of the series is a holomorphic function?

\(\mathbb{C}^*\) will denote \(\mathbb{C} \setminus \{0\}\), and \(D\) will denote the open unit disc in \(\mathbb{C}\); those \(z\)’s with \(|z| < 1\). \(L^p(D)\) is the set of measurable functions \(f\) on \(D\) with \(\int_D |f|^p \,dA\) finite (\(dA\) is area measure in the plane).

B3 For which \(p \in [1, \infty]\) and \(n \in \mathbb{Z}\) is \(z^n \in L^p(D)\)?

B4 For which \(p \in [1, \infty]\) is \(e^z \in L^p(D)\)?

B5 Can you find a function with an essential singularity at 0 so that the answer to the previous question is different?

B6 What are the Cauchy transforms of \(dx\) on \([0, 1]\), \(d\theta\) on \(\partial D\) (both of these are 1-dimensional Lebesgue measures), and the unit mass (the delta function) at 0? Note that \(\frac{1}{z^2} = O(|z|^{-1})\) as \(z \to \infty\) and is holomorphic in \(\mathbb{C}^*\). Is \(\frac{1}{z^2}\) the Cauchy transform of some measure?

B7 Use facts about Laurent series representations in \(\mathbb{C}^*\) to prove that \(\frac{\partial f}{\partial z} = g\) has a solution \(f \in C^\infty(\mathbb{C}^*)\) for every \(g \in C^\infty(\mathbb{C}^*)\).