Review Problems for the final exam in section 1 of Math 403

May 2, 2002

What follows is basically the final exam I gave last year in Math 403. I’ll happily discuss this exam with you on 

\{ 
Wednesday, May 8, 10 AM \\
Thursday, May 9, 3:30 PM
\} in Hill 525. I should also be available in my office (try Hill 304 first, then Hill 542) during much of the week, and accessible via e-mail most of the time. Please look over this semester’s earlier exams and review sheets.

(20) 1. In this problem \( U \) will be the region in the complex plane defined by the inequalities \( r = |z| > 1 \) and \( -\frac{\pi}{2} < \text{Arg} z < \frac{3\pi}{4} \).

a) Sketch the region \( U \) on the axes given. [Axes omitted here.]

Is \( U \) a connected open set? (Yes \| No) Is \( U \) a simply connected open set? (Yes \| No)

b) Suppose \( F(z) = z^2 \). If \( z = re^{i\theta} \), write a formula for \( F(z) \) in complex exponential form. If \( V = F(U) \), the image of \( U \) under \( F \), the collection of all values of \( F \) with domain restricted to \( U \), sketch the region \( V \) on the axes given. [Axes omitted here.] Is \( V \) a connected open set? (Yes \| No) Is \( V \) a simply connected open set? (Yes \| No)

c) Suppose \( G(z) = \frac{1}{z} \). If \( z = re^{i\theta} \), write a formula for \( G(z) \) in complex exponential form. If \( W = G(U) \), the image of \( U \) under \( G \), the collection of all values of \( G \) with domain restricted to \( U \), sketch the region \( W \) on the axes given. [Axes omitted here.] Is \( W \) a connected open set? (Yes \| No) Is \( W \) a simply connected open set? (Yes \| No)

(10) 2. Find \( \text{Arg} z \) if \( z = (1 + i)^i \).

(12) 3. Describe all solutions of \( z^3 = 2i \) algebraically in either rectangular or polar form. Sketch these solutions on the axes provided. [Axes omitted here.]

(18) 4. a) Suppose \( c(x, y) \) is a harmonic function, so \( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = 0 \).

Prove that \( f = \frac{\partial c}{\partial x} - i \frac{\partial c}{\partial y} \) is analytic.

b) Verify that \( w(x, y) = \cos x \cosh y \) is harmonic, and find one harmonic conjugate.

(25) 5. Compute \( \int_{\Gamma} e^{2\pi i z} \frac{z^2}{1} \, dz \) where \( \Gamma \) is the closed curve shown.

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6. a) If $k$ is a real number with $-1 < k < 1$, derive the Laurent series representation
\[
\frac{k}{z-k} = \sum_{n=1}^{\infty} \frac{k^n}{z^n} \quad (|k| < |z| < \infty).
\]
b) Write $z = e^{i\theta}$ in the equation obtained in part a) and then equate real parts on each side of the result to derive the summation formula
\[
\sum_{n=1}^{\infty} k^n \cos n\theta = \frac{k \cos \theta - k^2}{1 - 2k \cos \theta + k^2}.
\]

7. If $A$ is real and $A > 1$, Maple reports that
\[
\int_0^{\infty} \frac{x^2 \, dx}{(x^2 + 1)(x^2 + A^2)} = \left( \frac{1}{2} \right) \left( \frac{\pi}{A+1} \right).
\]
Check this assertion using the method of residues. Explain why some integral tends to 0 in the limit.

8. What is the radius of convergence of the Taylor series expansion of $h(z) = \frac{e^z}{(z-1)(z+2)}$ when expanded around $z = i$? Give a numerical answer. Justify why the series must converge with at least that radius and why it can’t have a larger radius.

Note Actual computation of the series is not practical.

9. Find the order of the pole of $H(z) = \frac{1}{(6\sin z + z^3 - 6z)^2}$ at $z = 0$.

10. Suppose $T$ is the inside of the square with corners $1+i, -1+i, -1-i, and 1-i$, and $S$ is the inside of the square with corners $3+3i, -3+3i, -3-3i, and 3-3i$. Suppose also that $f(z)$ is any entire function. Let $M$ be the maximum of $|f''(z)|$ on $T$ and let $N$ be the maximum of $|f(z)|$ on $S$. Show that $M \leq \frac{1}{2} N$.

Hint Begin by writing some complex variables formula connecting $f''$ and $f$.

11. Suppose $F(z) = z^3 \left( \frac{1}{z+1} + \frac{1}{(z-1)^3} \right)$, and $C$ is a simple closed curve which does not pass through 1 or $-1$. What are all possible values of $\int_C F(z) \, dz$ (and why)? Sketch examples of $C$’s which will give each value you list.

12. Find complex numbers $a, b, c,$ and $d$ so that the linear fractional transformation $L(z) = \frac{az+b}{cz+d}$ takes $-i$ to 0, 0 to 1, and $2i$ to $\infty$.

Sketch the image of the unit circle $|z| = 1$ under the transformation $L$, and briefly explain why what was drawn is correct.