Review Problems for the second exam in section 1 of Math 403

The review material is in three parts. This part contains the questions. Another page (available through the course homepage on the web) has bald answers (or hints, where appropriate) to the questions. Much more extended discussion of the answers is also available on the web through the course homepage. Students who may want a hint or a numerical answer can look at the bald answer page, and those desiring a more extended discussion can look for that. I hope this arrangement is helpful.

1. Suppose \( f(z) \) is analytic in the region \( 0 < |z| < 3 \), and suppose that \( f(\frac{1}{n}i) = 1 \) for all positive integers, \( n \). Explain why there must be a sequence \( \{z_n\} \) with \( \lim_{n \to \infty} z_n = 0 \) and so that \( \lim_{n \to \infty} f(z_n) \) exists and is 2.

2. Suppose \( g(z) = \frac{5}{z} + \frac{-3}{(z-1)^2} \), and that \( \gamma \) is the closed curve which is a triangle with vertices at \(-1, 2-i, \) and \(2+i\).
   a) What is \( \int_{\gamma} g(z) \, dz \)?
   b) What is \( \int_{\gamma} g'(z) \, dz \)?
   c) What is \( \int_{\gamma} (g(z))^2 \, dz \)?

3. Suppose that \( k(z) \) is an entire function satisfying the inequality \( |k(z)| \leq A \ln(|z|) + B \) for all \( z \neq 0 \). Prove that \( k(z) \) is constant.

4. Suppose that \( h(z) \) is an entire function, and that \( |h(z)| \leq |z^7| \) for all \( z \). Prove that \( h(z) = Cz^7 \) where \( C \) is a complex number with \( |C| \leq 1 \).

5. If \( m(z) = \frac{1}{z - \sin z} \), what is the type of the singularity of \( m(z) \) at 0? Find the first two non-zero terms of the Laurent series for \( m(z) \) at 0. What is the residue of \( m(z) \) at 0?

6. Suppose that \( P(z) = z^5 + 12z^2 + 2 \).
   a) Explain why \( P(z) \) has two roots (counted with multiplicity) inside the unit circle \( |z| = 1 \).
   b) Explain why \( P(z) \) has five roots (counted with multiplicity) inside the circle of radius 10 centered at 0.
   c) How many roots does \( P(z) \) have in the annulus \( 1 < |z| < 10 \)?

7. What is the radius of convergence of the Taylor series expansion centered at \( z = i \) of the function \( Q(z) = \frac{e^z}{(z-1)(z+1)(z-2)(z-3)} \)? Give an exact answer. Justify why the series must converge with at least that radius and why it can't have a larger radius. (Actual computation of the series is not practical and is not requested.)

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8. Suppose \( R(z) \) is the function defined by \( R(z) = \frac{z + 1}{z(z - 1)} \). Find the Laurent series representing \( R(z) \) in the annulus \( 0 < |z| < 1 \). Be sure to explain why the series you write is valid in that annulus. Find explicit values of the coefficients of \( z^{10} \) and \( z^{-10} \) in the series. Hint: Use partial fractions.

9. The function \( S(z) = (1 + 3z)e^{(z^2)} \) is analytic near 0.
   a) Use results about power series of familiar functions to find terms up to and including degree 4 in the Taylor series of \( S \) centered at \( z = 0 \).
   b) Use your answer to a) to compute \( S^{(4)}(0) \).

10. Use the Residue Theorem to compute \( \int_{0}^{2\pi} \frac{2}{3 + \sin \theta} d\theta \).

11. Use the Residue Theorem to compute \( \int_{0}^{\infty} \frac{dx}{\sqrt{x}(4 + x^2)} \).

12. Use the Residue Theorem to compute \( \int_{-\infty}^{\infty} \frac{\cos(ax)}{(1 + x^2)^2} dx \) if \( a \) is a positive real number.

13. Can there be a function \( q(z) \) which is analytic in some disc centered at 0 and which has the property that \( (q(z))^2 = z \) for all \( z \) in the disc?

14. What is the radius of convergence of the Taylor series centered at \( z = i \) for the function \( M(z) = \frac{\sin z}{z} \)?