Computing a definite integral

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Your goal here is to compute \( \int_{-\infty}^{\infty} \frac{1}{x^2 + x + 1} \, dx \) using the Residue Theorem.

1. We may write \( \int_{-\infty}^{\infty} \frac{1}{x^2 + x + 1} \, dx = \lim_{R \to \infty} \int_{I_R} f(z) \, dz \) where \( I_R \) is the interval from \(-R\) to \(R\) on the real line. \( f \) is an analytic function with two isolated singularities. It is given by a formula:

\[
f(z) = \quad \text{.}
\]

2. The isolated singularities of \( f \) are at the complex numbers \( A = \quad \text{ in the upper half plane and } B = \quad \text{ in the lower half plane.} \)

The type of the isolated singularity at \( A \) is (circle one)

A POLE A REMOVABLE SINGULARITY AN ESSENTIAL SINGULARITY

3. We can factor the denominator of \( f \) and rewrite its formula as follows:

\[
f(z) = \quad \text{.}
\]

4. The function \( f \) has an isolated singularity at \( A \) in the upper half plane. Since \( f \) has a \quad \text{(insert here a useful and precise descriptive phrase about \( f \)'s singularity at \( A \))}, the residue of \( f \) at \( A \) is easy to compute.

That residue is \quad \text{.}

5. Suppose \( S_R \) is the counterclockwise oriented semicircle of radius \( R \) in the upper half plane centered at 0. We will write \( I_R + S_R \) to mean the simple closed curve obtained by following \( I_R \) by \( S_R \). If \( R \) is large enough (as shown) so \( A \) is inside \( I_R + S_R \), the Residue Theorem tells us that

the value of \( \int_{I_R + S_R} f(z) \, dz \) is \quad \text{.}
6 If $|z| = R$ where $R$ is a large positive number, then the “reverse triangle inequality” applied to the original formula for $f$ allows us to underestimate the denominator of $|f(z)|$ in terms of $R$ (note that there will be several expressions that are subtracted):

So the denominator of $|f(z)| \geq \ldots$.

7 The length of $S_R$ is $\ldots$. The ML inequality allows us to overestimate the modulus of $\int_{S_R} f(z) \, dz$ in terms of $R$.

$$\left| \int_{S_R} f(z) \, dz \right| \leq \ldots$$

8 We therefore conclude that $\lim_{R \to \infty} \int_{S_R} f(z) \, dz = \ldots$.

9 Combine the results of 8 and 5 to compute the value of $\lim_{R \to \infty} \int_{I_R} f(z) \, dz$.

The value of this limit is $\ldots$.

10 Putting it all together, we finally can write

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + x + 1} \, dx = \ldots$$

Maple reports that this quantity is approximately 3.62759 87284 68435 7012.