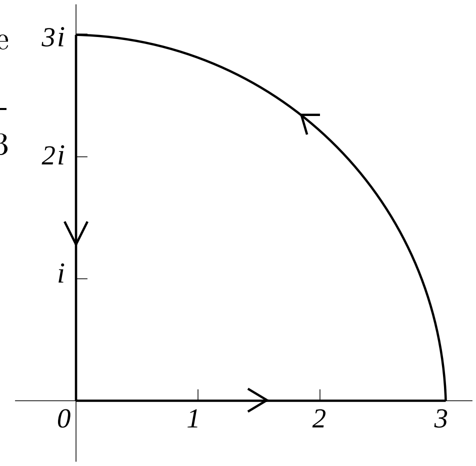
(20) 1. Compute $\int_B \frac{z^4}{(z-(1+i))^3}\,dz$ where B is the simple closed curve shown: the line segment from 0 to 3, followed by the quarter-circular arc centered at 0 from 3 to $3i$, followed by the line segment from $3i$ to 0.



Answer -24π

(20) 2. Use the Residue Theorem to compute $\displaystyle\int_0^\infty \frac{\sqrt{x}}{x^2+4}\,dx$.

Comment `Maple` reports that the answer is $\frac{\pi}{2}$.

You must show details of any estimates required to apply the Residue Theorem to earn full credit here, in addition to the computation of any residues necessary.

(20) 3. Suppose $H(z) = \frac{1}{\sin z} - \frac{1}{z}$.
Identify as precisely as possible the type of the isolated singularity at 0 of $H(z)$: is it removable, a pole, or essential? If it is a pole, find the order of the pole.
Find the first two non-zero terms of the Laurent series of $H(z)$ at 0.
Find the residue of $H(z)$ at 0.

Type of singularity __________

First two terms __________

Residue at 0 __________

(20) 4. Show that $u(x,y) = xy + y$ is a harmonic function and find a harmonic conjugate $v(x,y)$ of $u(x,y)$. Write the resulting complex analytic function $u(x,y) + iv(x,y)$ in terms of the complex variable $z = x + iy$.

(20) 5. Suppose $G(z) = z^5 + 5z^2 + e^z$. How many zeros (counting multiplicity) does G have in the annular region $1 < |z| < 2$?

(20) 6. a) Use geometric series to find a simple expression, $S(x)$, for $\displaystyle\sum_{n=1}^\infty \frac{e^{inx}}{2^n}$ when x is a real number, and explain briefly why the series converges.

b) Take imaginary parts of both sides of the equation $S(x) = \displaystyle\sum_{n=1}^\infty \frac{e^{inx}}{2^n}$ and verify that

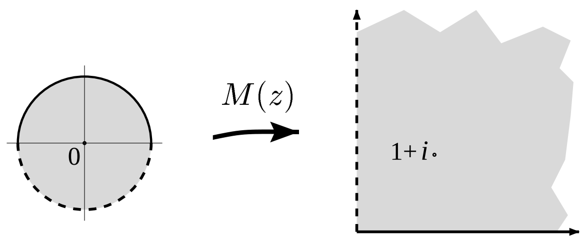
$$\frac{2\sin x}{5 - 4\cos x} = \sum_{n=1}^\infty \frac{\sin(nx)}{2^n}.$$

(20) 7. Here D is the open unit disc in the complex plane: z's with $|z| < 1$, and Q is the open first quadrant in the complex plane: z's with $\operatorname{Re} z > 0$ **and** $\operatorname{Im} z > 0$.

Find a conformal mapping $M(z)$ from D onto Q so that

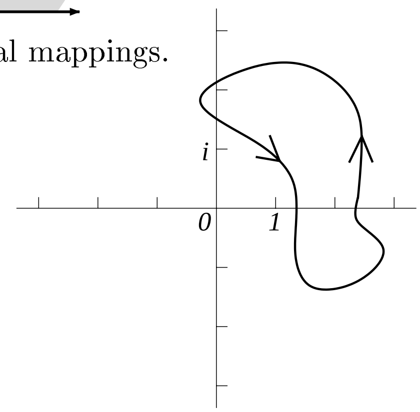
- $M(0) = 1 + i$.
- $M(z)$ extends continuously to map the upper semicircle ($|z| = 1$ **and** $\operatorname{Im} z > 0$) to the positive real axis.
- $M(z)$ extends continuously to map the lower semicircle ($|z| = 1$ **and** $\operatorname{Im} z < 0$) to the positive imaginary axis.

These requirements are indicated in the picture below.



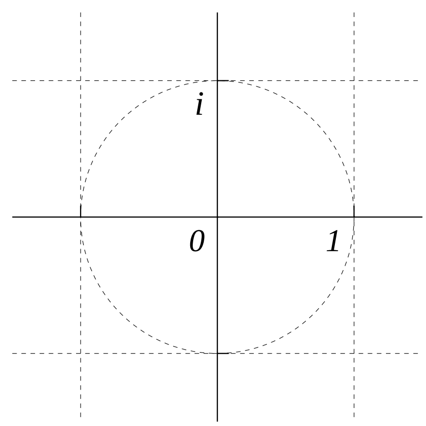
Hint Write $M(z)$ as a composition of two simpler conformal mappings.

(20) 8. This problem has two parts with equal values.



a) Compute $\int_C \operatorname{Log} z \, dz$ where C is the curve shown. Briefly explain your answer.

b) Describe all solutions of $z^4 = -1$ algebraically in rectangular form. Sketch these solutions on the axes provided.



Comment Your answer(s) should be exact. Answers may use traditional mathematical constants such as π and e and operations involving arithmetic and root extraction of positive real numbers.

(20) 9. Find the radius of convergence of the Taylor series expansion centered at $z = i$ of the function $W(z) = \dfrac{e^z - 1}{z(z-1)}$. Give an exact answer. Justify why the series must converge with at least that radius <u>and</u> why it can't have a larger radius. (Actual computation of the series is not practical and is not requested.)

Comment Watch for the tricky part! Don't ass u me ...

(20) 10. Suppose $F(z)$ is an analytic function defined in an open disc centered at 0 and

$$|F^{(k)}(0)| \leq 7$$

for every positive integer k. Explain why $F(z)$ must actually be an entire function: it can be extended to a function defined and analytic for all z in $\mathbb{C}$.

Extra credit (5 points) Suppose that $F(z)$ is defined in a connected open set containing 0 with the same estimates on the derivatives. Explain why the same conclusion still holds, and explain why the word "connected" is needed.

Final Exam for Math 403, section 1

May 12, 2002

NAME ______________________________

Do all problems, in any order.
Show your work. An answer alone may not receive full credit.
No notes or calculators may be used on this exam.

Problem Number	Possible Points	Points Earned:
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total Points Earned:		