1. Use the Residue Theorem to compute \( \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} \).

**Answer** We apply the Residue Theorem to the function \( f(z) = \frac{1}{(1+z^2)^2} \) on the simple closed curve \( I_R + S_R \), where \( I_R = [-R, R] \) is an interval on the real axis, and \( S_R \) is the upper semicircle: \( |z| = R \) and \( \text{Im} \ z \geq 0 \). The curve is oriented counterclockwise, and \( R > 1 \). \( f(z) \) has isolated singularities at \( \pm i \), and since \( f(z) = \frac{1}{(z-i)^2(z+i)^2} \), the isolated singularities are poles of order 2. The singularity at \( i \) is inside the closed curve. If we write \( f(z) = \frac{H(z)}{(z-i)^2} \), then the residue of \( f(z) \) at \( z = i \) is just \( H'(i) \) since \( H(z) = H(i) + H'(i)(z-i) + \text{higher order terms} \). Here \( H(z) = \frac{1}{(z+i)^2} \) so \( H'(z) = \frac{-2z}{(z+i)^3} \) and \( H'(i) = \frac{1}{4i} \). As \( R \to \infty \), \( \int_{I_R} f(z) \, dz \to \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} \).

If \( |z| = R \), \( |1 + z^2| \geq R^2 - 1 \) so that for \( R > 1 \), \( \left| \int_{S_R} f(z) \, dz \right| \leq \pi R \cdot \frac{1}{(R^2-1)^2} \) by the ML inequality. Therefore as \( R \to \infty \), \( \int_{S_R} f(z) \, dz \to 0 \). When \( R > 1 \), the Residue Theorem shows that \( \int_{I_R + S_R} f(z) \, dz = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2} \). As \( R \to \infty \), we see that \( \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2} \).

2. Find \( R > 0 \) so that all the roots of the polynomial \( P(z) = z^5 + 12z^4 - (1+i)z^2 + 9z - 3 \) are inside the circle \( |z| = R \).

**Answer** Take \( R = 100 \), \( f(z) = z^5 \), and \( g(z) = 12z^4 - (1+i)z^2 + 9z - 3 \). For \( |z| = 100 \), \( \Delta f(z) = 10^{10} \) while \( |g(z)| \leq 12|z|^4 + \sqrt{2}|z|^2 + 9|z| + 3 \leq 10^{8}(12 + \sqrt{2} + 9 + 3) < 10^{10} \). We use Rouché’s Theorem: \( f(z) \) and \( g(z) = P(z) \) must have the same number of zeros inside the circle \( |z| = 100 \). But \( f(z) \) has a zero of multiplicity 5 at 0. So \( P(z) \) must have five zeros inside \( |z| = 100 \) and \( P(z) \), a polynomial of degree 5, can have at most five zeros.

3. Use the Residue Theorem to compute \( \int_0^{2\pi} \frac{d\theta}{\cos^3 \theta} \).

**Answer** Since \( \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \), \( \int_0^{2\pi} \frac{d\theta}{\cos^3 \theta} = \int_0^{2\pi} \frac{d\theta}{10e^{i\theta} + 3(e^{i\theta})^3} \). If \( z = e^{i\theta} \) so \( dz = ie^{i\theta} d\theta \), the definite integral is recognized as a parameterization of a line integral around the unit circle, \( |z| = 1 \): \( \frac{1}{i} \frac{2}{3z^2 + 1 + 2} \, dz \). \( 3z^2 + 10z + 3 = 0 \) when \( z = \frac{-10 \pm \sqrt{100 - 36}}{6} = \frac{-1 \pm 2}{3} \). The integrand has isolated singularities which are both poles of order 1 at \(-\frac{1}{3} \) and \( \frac{1}{3} \). To use the Residue Theorem, we need the residue inside \( |z| = 1 \). Since \( \frac{2}{3z^2 + 10z + 3} = \frac{2}{3(z + \frac{3}{2})} \), the residue of the integrand at \( -\frac{1}{3} \) is \( \frac{2}{3(z + \frac{3}{2})} \) at \( z = -\frac{1}{3} \), which is \( -\frac{1}{3} \). Then the Residue Theorem gives the value of the desired integral: \( 2\pi i \cdot -\frac{1}{3} \cdot \frac{2}{3} = \frac{\pi i}{3} \).

4. Suppose \( Q(z) = \frac{(e^z - 1)^2}{z^4} \). Identify as precisely as possible the type of the isolated singularity at 0 of \( Q(z) \): is it removable, a pole, or essential? If it is a pole, find the order of the pole. Find the first two non-zero terms of the Laurent series of \( Q(z) \) at 0. Find the residue of \( Q(z) \) at 0.

**Answer** We know \( e^z = 1 + z + \frac{z^2}{2} + \text{higher order terms} \) so that \( (e^z - 1)^2 = \frac{(z + \frac{z^2}{2} + \text{h.o.t.})^2}{z^4} = \frac{z^2 + z^3 + \text{h.o.t.}}{z^4} = \frac{1}{z} + \frac{1}{z} + \text{h.o.t.} \) and we can read off the answers: \( Q(z) \) has a pole of order 2 at 0, its Laurent series begins \( \frac{1}{z^2} + \frac{1}{z} \), and its residue at 0 is 1.

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5. Suppose $f(z)$ is an entire function (analytic in the whole plane) and that $|f(z)| \leq |e^z|$ for all complex numbers $z$. Show that there must be a complex number $C$ with $|C| \leq 1$ so that $f(z) = Ce^z$ for all $z$.

**Answer** Since $e^z$ is never 0, $F(z) = \frac{f(z)}{e^z}$ is an entire function. The hypotheses say that $|F(z)| \leq 1$ for all $z$. Liouville's Theorem implies that $F(z)$ is constant, and it must be a constant $C$ with $|C| \leq 1$. So $\frac{f(z)}{e^z} = C$ for all $z$, and $f(z) = Ce^z$ for all $z$, as desired.

6. a) Compute $\int_{|z|=1} \frac{e^z}{z} \, dz$ using any applicable theorem.

**Answer** We could use the Residue Theorem again, but the Cauchy Integral Formula for $z = 0$ also applies. The result is $2\pi i$ multiplied by the value of $e^z$ at $z = 0$. This value is just 1, so the integral’s value is $2\pi i$.

b) Use the answer given for part a) to find the exact values of $\int_{0}^{2\pi} e^{i \cos \theta} \cos (\sin \theta) \, d\theta$ and of $\int_{0}^{2\pi} e^{i \cos \theta} \sin (\sin \theta) \, d\theta$.

**Answer** If $z = e^{i \theta}, e^z = e^{(e^{i \theta})} = e^{i \cos \theta + i \sin \theta} = e^{i \cos \theta} e^{i \sin \theta} = e^{i \cos \theta} (\cos(\sin \theta) + i \sin(\sin \theta))$. Also $dz = ie^{i \theta} \, d\theta$ so that $\frac{dz}{z} = \frac{ie^{i \theta} \, d\theta}{e^{i \theta}} = i d\theta$. Therefore $\int_{|z|=1} \frac{e^z}{z} \, dz = \int_{0}^{2\pi} e^{i \cos \theta} (\cos(\sin \theta) + i \sin(\sin \theta)) i \, d\theta$. Part a) tells us this is $2\pi i$. We see that $\int_{0}^{2\pi} e^{i \cos \theta} \cos (\sin \theta) \, d\theta = 2\pi$ and $\int_{0}^{2\pi} e^{i \cos \theta} \sin (\sin \theta) \, d\theta = 0$ by separating the real and imaginary parts. This is consistent with Maple’s approximations.

**Comment** The versions of Maple I have at home can’t compute this integral symbolically, but the latest version at Rutgers can.

7. The following information is known about a function, $F(z)$:

   i) $F(z)$ is defined and analytic for all $z \neq 0$.
   ii) $F(i) = 3$.
   iii) For all positive integers, $n$, $F(\frac{1}{n}) = 0$.

a) What kind of isolated singularity must $F(z)$ have at 0? Explain your answer.

**Answer** If 0 were a pole then $|F(\frac{1}{n})| \to \infty$ as $n \to \infty$ and this is false by iii). If 0 were a removable singularity, then $F(0) = \lim_{n \to \infty} F(\frac{1}{n}) = 0$. If this is true, $F(z)$ with the value 0 at $z = 0$ would be entire. Then $F(z)$ would be 0 on a sequence with a limit point. Such a function would need to be 0 everywhere by the Identity Theorem (“Two functions analytic in a connected open set which agree on a set with a limit point must actually agree everywhere in the set.”), but this would contradict ii), that $F(i) = 3$. The only alternative is that $F(z)$ has an essential singularity at $z = 0$.

b) What is the radius of convergence of the Taylor series expansion centered at $z = i$ of the function $F(z)$? Explain your answer.

**Answer** The radius of convergence is at least 1, because $F(z)$ is certainly analytic in a disc of radius 1 centered at $i$. If the radius of convergence were larger than 1, the sum would represent an analytic function agreeing with $F(z)$ in an open disc, and therefore agreeing with $F(z)$ for all $z \neq 0$ in the disc (using the Identity Theorem again). Then $F(z)$ would have a removable singularity at 0 because the Taylor series would behave like an analytic function near 0. Since $F(z)$ has an essential singularity at 0, this is impossible. So the radius of convergence must be exactly 1.