1. Find the principal argument \( \text{Arg} z \) when \( z = (2 - 2i)^7 \).

**Answer** \( 2 - 2i \) is in the fourth quadrant, and one value if its argument is \(-\frac{\pi}{4}\). One value of the argument of its seventh power is \(-\frac{7\pi}{4}\). The “Arg” of that number must be in the interval between \(-\pi\) and \(\pi\) (including \(\pi\)), so \( \text{Arg } z = \frac{5\pi}{4} \).

Alternatively, \( z = 2e^{-i\pi/4} \) so \( z^7 = 2^7 e^{-7i\pi/4} \), leading to the same answer.

2. Describe all solutions of \( z^5 = 1 \) algebraically in either rectangular or polar form. Sketch these solutions on the axes provided.

**Answer** If \( z^5 = 1 \) and \( z = re^{i\theta} \), then \( z^5 = r^5e^{5i\theta} \). If \( r^5 = 1 \), with \( r \) a non-negative real, then \( r = 0 \) (mod 2\( \pi \)), which leads to the list \( z = e^{2\pi ik/5} = \cos(2\pi k/5) + i\sin(2\pi k/5) \) for integers \( k \) in the set \( \{0, 1, 2, 3, 4\} \). There are “simple” rectangular expressions for these complex numbers involving square roots of small integers. I don’t know them.\(^*\) The picture I want will indicate the vertices of a regular pentagon inscribed in the unit circle with one vertex at 1. The central angle between the vertices is \( 2\pi/5 \) or 72°.

\[ \text{Diagram} \]

3. Suppose \( f \) is an analytic function with real part \( u(x, y) \) and imaginary part \( v(x, y) \). Explain why the function \( H(x, y) = (u(x, y))^2 - (v(x, y))^2 \) is harmonic, and find a harmonic conjugate of \( H(x, y) \) in terms of \( u(x, y) \) and \( v(x, y) \).

**Answer** \( Q(z) = z^2 = x^2 - y^2 + i(2xy) \) is analytic, and since composition of analytic functions is analytic, \( Q(f(z)) \) is analytic. The real part of \( Q(f(z)) \) is \( H(x, y) \), and since the real part of an analytic function is harmonic, \( H(x, y) \) is harmonic. The imaginary part of \( Q(f(z)) \) will be a harmonic conjugate of \( H(x, y) \), and this is \( 2u(x, y)v(x, y) \).

An answer can also be obtained through a direct computation of the Laplacian of \( H \), use of the Cauchy-Riemann equations, etc., but the computation is elaborate, allowing many opportunities for error. Several very persistent students solved the problem this way correctly. A solution might go like this:

\[ \begin{align*}
\text{First derivatives} & \quad \text{Second derivatives} & \quad \text{Use of the Cauchy-Riemann equations} \\
H_x = 2uu_x - 2v v_x & \quad H_{xx} = 2u u_{xx} + 2u u_x - 2v_y v_x - 2v_{xxx} & \quad H_{xx} = 2u u_x + 2u u_{xx} - 2u u_y - 2v_{xx} \\
H_y = 2uu_y - 2v v_y & \quad H_{yy} = 2u u_y + 2u u_y - 2v_y v_y - 2v_{yy} & \quad H_{yy} = 2u u_y + 2u u_{yy} - 2u u_x - 2v_{yy}
\end{align*} \]

Then add, and notice that since \( u \) and \( v \) are both harmonic (so \( u_{xx} + v_{yy} = 0 \) and \( u_{xx} + v_{yy} = 0 \)) that \( H_{xx} + H_{yy} = 0 \). Some further careful effort is needed to find a harmonic conjugate of \( H \). If \( K \) is such a function, then \( K_x = -H_y = -(2uu_y - 2w_y) \) and \( K_y = H_x = 2uu_x - 2v_x \) so again using the Cauchy-Riemann equations, \( K_x = -(2uu_x - 2v_x) = 2uu_x + 2uu_x \) and \( K_y = 2uu_y - 2v(-u_y) = 2uu_y + 2uu_y \). An inspired guess (?) shows that \( K = 2uv \) satisfies these equations.

4. a) Suppose \( S \) is the rectangular region whose boundary is the rectangle with corners 0, 1, \( \pi i \), and 1 + \( \pi i \) as shown. Sketch the image of \( S \) under the exponential mapping on the axes given.

**Answer** The modulus of \( e^z \) is \( e^{\text{Re } z} \), so that the image of \( S \) under the exponential map varies from \( e^0 \) to \( e^1 \approx 27 \). The argument of \( e^z \) is \( \text{Im } z \), so the argument varies from 0 to \( \pi \).

The result is the half-annulus shown.

b) Find all solutions of \((e^z)^2 = -1\).

**Answer** Look for \( z \) so that \( e^z = i \) or \( e^z = -i \). If \( e^z = i \), then \( z = \log i = \ln |i| + i \text{arg } i \) so \( z \) must be \( i(\frac{\pi}{2} + 2\pi k) \) for any integer \( k \). One solution (with \( k = 0 \)) can be seen in the answer for the previous part. Similarly, the solutions of \( e^z = -i \) are \( i(\frac{3\pi}{2} + 2\pi k) \) for any integer \( k \).

\[ \text{OVER} \]

\(^*\) Since \( z^5 - 1 = (z - 1)(z^4 + z^3 + z^2 + z + 1) \), one root is 1. Maple reports that the other roots are \( \frac{1}{4}\sqrt{5} - \frac{1}{4} + \frac{1}{4}i\sqrt{2}\sqrt{5 + \sqrt{5}, \frac{1}{4}\sqrt{5} - \frac{1}{4} + \frac{1}{4}i\sqrt{2}\sqrt{5 - \sqrt{5}, \frac{1}{4}\sqrt{5} - \frac{1}{4} - \frac{1}{4}i\sqrt{2}\sqrt{5 - \sqrt{5}}, \text{and } \frac{1}{4}\sqrt{5} - \frac{1}{4} - \frac{1}{4}i\sqrt{2}\sqrt{5 + \sqrt{5}.} \]
5. Find the two points in the complex plane where the function \((\frac{1}{4}x^4 + y^2) + i(x^2y)\) is complex differentiable. Where is this function analytic?

**Answer** The function is complex differentiable where the Cauchy–Riemann equations are satisfied. Here \(u(x, y) = \frac{1}{4}x^4 + y^2\) and \(v(x, y) = x^2y\). \(u_x = v_y\) becomes \(x^3 = y^2\) and solutions of this are \(x = 0\) and \(x = 1\). \(u_y = -v_x\) is \(2y = -2xy\) here. When \(x = 0\) we see that \(y = 0\). When \(x = 1, y = 0\). The two points are 0 and 1. The function is not complex differentiable in any open set so it is analytic nowhere.

6. Describe with some justification what happens to \(\sin z\) as \(z \to \infty\) along the positive imaginary axis.

**Answer** \(\sin z = \sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy)\). When \(x = 0\) the equation becomes \(\sin(iy) = i\sinh(y)\). As \(y \to +\infty\), \(\sinh(y)\) increases steadily without any upper bound. So on the positive imaginary axis as \(y \to +\infty\), \(\sin z\) is imaginary, increasing, and unbounded.

7. Describe a connected open set \(W\) of the plane that includes the positive real axis on which an analytic branch of \(\log(z^3 + 5)\) can be defined. Justify your assertion with an explanation. **Hint** A simple wedge might be a good candidate.

**Answer** \(\log z\) is analytic in the complex plane with the non-positive real axis deleted. If we can find a domain for \(z^3 + 5\) which maps into that part of the plane and includes the positive real axis we will be done. For example, consider the “wedge” \(W\) defined by requiring that \(\text{Arg } z\) be greater than \(-\frac{\pi}{6}\) and less than \(\frac{\pi}{3}\). This is a connected open set containing the positive real numbers. Then cubing maps that wedge into the set with \(\text{Arg } z\) between \(-\frac{\pi}{2}\) and less than \(\frac{\pi}{3}\). Adding 5 to points in this set moves them to the right 5 units – still in the set. And the result is inside the domain where \(\log z\) is analytic.

Starting with \(W\) The effect of cubing Right translation by 5 Sitting inside \(\log z\)’s domain

8. a) Compute \(\int_C (x^2 + iy) \, dz\) where \(C\) is the straight line segment from 0 to 1 + 2i.

**Answer** Parameterize by \(z = x + iy = (1 + 2i)t\) with \(0 \leq t \leq 1\). Then \(dz = (1 + 2i) \, dt\) and \(x = t\) and \(y = 2t\), so that \(x^2 + iy = t^2 + 2it\). The contour integral becomes \(\int_0^1 ((t^2 + 2it)(1 + 2i) \, dt = (\frac{1}{3} + i)(1 + 2i)\).

This answer is fine. If you must “simplify”, you should get \(-\frac{\pi}{3} + \frac{2\pi}{3}i\).

b) If \(C\) is the circle of radius 2 centered at 0, verify that \(\left|\int_C \left(\frac{z^2 + 3}{5z^3 - 3}\right) \, dz\right| \leq \frac{28\pi}{37}\). Show details of your estimates.

**Answer** We overestimate the modulus of the integral by \(ML\) where \(L\), the length, is \((2\pi)\text{radius} = (2\pi)2 = 4\pi\), and \(M\) is an overestimate of the integrand on the circle. The integrand is a quotient. We estimate it by first overestimating the top: \(|z^2 + 3| \leq |z|^2 + 3 = 2^2 + 3 = 7\) on the circle. We next underestimate the bottom: \(|5z^3 - 3| = |5z^3 + (-3)| \geq |5|z|^3| - | - 3| = 5|z|^3 - 3 = 5 \cdot 2^3 - 3 - 37\) on the circle. Therefore the modulus of the integrand is estimated by \(\frac{7}{37}\) and \(ML = \frac{7}{37} \cdot 4\pi = \frac{28\pi}{37}\) as desired.

9. Compute \(\int_C \left(\frac{7}{z^3} + \frac{4}{z} + 5z^8\right) \, dz\) where \(C\) is the boundary of the unit circle.

**Answer** We use linearity of the integral and evaluate the three parts separately. \(\frac{7}{z^3}\) has an analytic antiderivative: use \(-\frac{7}{2}z^{-2}\) in the “punctured plane” (that is, the complex numbers except for 0). Therefore its integral around the closed curve \(C\) must be 0. The same is true for \(5z^8\), where the antiderivative is \(\frac{5}{9}z^9\). What about \(\int_C \frac{4}{z} \, dz\)? This is 4 times the value of \(\int_C \frac{1}{z} \, dz = 2\pi i\). We computed this in class and it is also in the textbook. Thus the combined value of the integrals must be \(4 \cdot 2\pi i = 8\pi i\).

This problem can also be done directly using a parameterization. Take \(z = e^{i\theta}\) so that \(dz = ie^{i\theta} \, d\theta\) with \(0 \leq \theta \leq 2\pi\). Then \(\frac{7}{z^3} + \frac{4}{z} + 5z^8\) becomes \(7e^{-7i\theta} + 1e^{-i\theta} + 5e^{8i\theta}\). The whole integral is \(\int_0^{2\pi} (7e^{-7i\theta} + 4e^{-i\theta} + 5e^{8i\theta}) \, i \, d\theta\), which can be evaluated using the Fundamental Theorem of Calculus. The \(2\pi\)-periodicity of the exponential function implies that the first and third terms result in 0. The middle term is the integral of a constant on the interval from 0 to \(2\pi\). It results in \(4 \cdot 2\pi \cdot i = 8\pi i\).