Please write solutions to two of these problems. Hand them in next Wednesday, February 3. These written solutions should be accompanied by explanations using complete English sentences. In this case, all students must work in groups: a group is at least two students and at most four students (recommended size of the group: three). All students should contribute to the writeup and should sign what is handed in.

Definition Suppose $\mathcal{P}(n)$ is a statement involving a positive integer, n. Then we will write "$\mathcal{P}(n)$ is true for all n which is sufficiently large" if there exists $n_0 \in \mathbb{N}$ so that $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$ with $n \geq n_0$.

Comment Please note that when working with "sufficiently large" assertions you are <u>not</u> asked to make a "best possible" assertion, that is, find the smallest valid n_0. You just need to find some n_0 that works! You must be sure, however, to include some specific n_0 in your argument.

1. $n!$ grows very fast.

a) Prove that $n^3 \leq n!$ for all $n \in \mathbb{N}$ which is sufficiently large.

b) Prove that $n^7 \leq n!$ for all $n \in \mathbb{N}$ which is sufficiently large.

c) Suppose $k \in \mathbb{N}$ is fixed. Prove that $n^k \leq n!$ for all $n \in \mathbb{N}$ which is sufficiently large.

2. $n!$ grows very fast.

a) Prove that $3^n \leq n!$ for all $n \in \mathbb{N}$ which is sufficiently large.

b) Prove that $7^n \leq n!$ for all $n \in \mathbb{N}$ which is sufficiently large.

c) Suppose $k \in \mathbb{N}$ is fixed. Prove that $k^n \leq n!$ for all $n \in \mathbb{N}$ which is sufficiently large.

3. Compare the rates of growth of n^k and k^n for $k \in \mathbb{N}$ fixed and $n \in \mathbb{N}$. Which grows faster? Make your assertion precise and prove it. Suggestion: try some special cases first, like $k = 3$ and $k = 7$. You will need to prove "sufficiently large" statements here, too.

4. a) Suppose a and b are positive numbers. "Expand" $(a-b)^2$ (what do you know about squares and order?) and prove that $a^2 + b^2 \geq 2ab$.

b) Suppose a and b are positive numbers. Prove that $\frac{a}{b} + \frac{b}{a} \geq 2$.

c) Suppose a and b are positive numbers. Prove that $(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$.

d) Suppose a and b and c are positive numbers. Prove that $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$.

e) Suppose that n is a positive integer, and that $a_1, a_2, \ldots, a_n$ are all positive numbers. Prove that $\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} \frac{1}{a_j}\right) \geq n^2$.

5. If a particular $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the following two conditions:

I) For all $x, y \in \mathbb{R}$, $f(xy) = f(x)f(y)$. II) For all $x \in \mathbb{R}$ with $x \neq 0$, $f(x) \neq 0$.

then f must satisfy the condition

III) For all $x, y \in \mathbb{R}$ with $y \neq 0$, $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$.