Formula sheet for the final exam in Math 291, spring 2003

FIRST VERSION 3/2/2003: CORRECTED FROM FALL 2002; AMENDED 3/13/2003; INCREASED 5/5/2003

Cauchy-Schwarz: $|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\,\|\mathbf{w}\|$. Triangle inequality: $\|\mathbf{v}+\mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$.

Distance from $P_0(x_0,y_0,z_0)$ to $P_1(x_1,y_1,z_1)$ is $\sqrt{(x_0-x_1)^2+(y_0-y_1)^2+(z_0-z_1)^2}$.

Distance from $P_1(x_1,y_1,z_1)$ to the plane $ax+by+cz=d$ is $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$.

Sphere: $(x-h)^2+(y-k)^2+(z-l)^2=r^2$

Plane: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ where $\mathbf{n}=\langle a,b,c\rangle$

Line: $\{x=x_0+at, y=y_0+bt, z=z_0+ct\}$ through (x_0,y_0,z_0) in direction $\langle a,b,c\rangle$

$\|\mathbf{a}\| = \sqrt{(a_1)^2+(a_2)^2+(a_3)^2}$

$|\mathbf{a}\cdot\mathbf{b}| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta$ (If $=0$, then $\mathbf{a}\perp\mathbf{b}$.) $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\sin\theta$ (If $\mathbf{a}\|\mathbf{b}$, this $=0$.)

$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ $\quad \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ $\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$\text{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$ $\quad \text{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$

Volume of a parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$: $\|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\|$

Arc length: $\int_a^b \|\mathbf{r}'(t)\|\,dt$ $\quad \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ $\quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ $\quad \mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$

$$\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \overset{2\ \text{dim}}{=} \frac{|y''(t)x'\,(t)-x''(t)y'(t)|}{(x'(t)^2+y'(t)^2)^{3/2}} \overset{y=f(x)}{=} \frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$$

$\tau = \dfrac{(\mathbf{r}'(t)\times\mathbf{r}''(t))\cdot\mathbf{r}'''(t)}{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|^2}$. $\quad$ Frenet-Serret: $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, $\dfrac{d\mathbf{N}}{ds} = -\kappa\mathbf{T}+\tau\mathbf{B}$, $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$.

Tangent plane to $z=f(x,y)$ at $P(x_0,y_0,z_0)$: $z-z_0 = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

Linear approximation to $f(x,y)$ at (a,b): $f(x,y)\approx f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$

Tangent plane to $F(x,y,z)=0$:

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

If y implicitly defined by $y=f(x)$ in $F(x,y)=0$ then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

If z implicitly defined by $z=f(x,y)$ in $F(x,y,z)=0$ then $z_x = -\dfrac{F_x}{F_z}$ and $z_y = -\dfrac{F_y}{F_z}$.

$\nabla f = \langle f_x, f_y, f_z\rangle = \dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}+\dfrac{\partial f}{\partial z}\mathbf{k}$ $\quad D_{\mathbf{u}}f(x,y,z) = \nabla f(x,y,z)\cdot\mathbf{u}$

Some chain rules:

If $z=f(x,y)$ and $x=x(t)$ and $y=y(t)$, then $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

If $z=f(x,y)$ and $x=g(s,t)$ and $y=h(s,t)$, then $\dfrac{\partial z}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial s}+\dfrac{\partial f}{\partial y}\dfrac{\partial h}{\partial s}$.

Suppose $f_x(a,b)=0$ and $f_y(a,b)=0$. Let $H=H(a,b)=f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$.

a) If $H>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.

b) If $H>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.

c) If $H<0$, then $f(a,b)$ is not a local maximum or minimum (f has a saddle point).

A real-valued function $F(\mathbf{x})$ is continuous at $\mathbf{x_0}$ if, given any $\varepsilon>0$, there is a $\delta>0$ so that whenever $\|\mathbf{x}-\mathbf{x_0}\|<\delta$, then $|F(\mathbf{x})-F(\mathbf{x_0})|<\varepsilon$.

Lagrange multipliers for one constraint

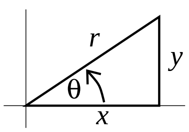
If G(the variables) = a constant is the constraint and we want to extremize the objective function, F(the variables), then the extreme values can be found among F's values of the solutions of the system of equations $\nabla G = \lambda \nabla F$ (a vector abbreviation for the equations $\lambda \frac{\partial F}{\partial \star} = \frac{\partial G}{\partial \star}$ where $\star$ is each of the variables) **and** the constraint equation.

Polar coordinates

$x = r\cos\theta \quad y = r\sin\theta$

$r^2 = x^2 + y^2 \quad \theta = \arctan\left(\frac{y}{x}\right)$

$dA = r\,dr\,d\theta$

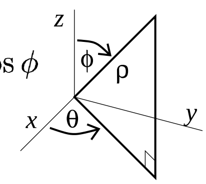


Spherical coordinates

$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$

$\rho^2 = x^2 + y^2 + z^2$

$dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$



Change of variables in 2 dimensions

$\int\int_R f(x,y)\,dA = \int\int_{\tilde{R}} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$; $\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$, the Jacobian.

Total mass of a mass distribution $\rho(x,y,z)$ over a region R of $\mathbb{R}^3$ is $\int\int\int_R \rho(x,y,z)\,dV$.

Line integral formulas

$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$

$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b P(x(t),y(t))x'(t)\,dt + Q(x(t),y(t))y'(t)\,dt$

Green's Theorem

$\int_C P\,dx + Q\,dy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

These P, Q pairs will give R's area $\begin{cases} P=-y \text{ and } Q=0 \\ P=0 \text{ and } Q=x \\ P=-\frac{1}{2}y \text{ and } Q=\frac{1}{2}x \end{cases}$

A **conservative vector field** $\mathbf{V} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a **gradient vector field**: there's $f(x,y)$ with $\nabla f = \mathbf{V}$ so $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$. f is a **potential** for $\mathbf{V}$. A conservative vector field is **path independent**. Work done by such a vector field over a **closed curve** is 0. For V conservative with potential f: $\int_C P\,dx + Q\,dy = f(\text{THE END}) - f(\text{THE START})$.

If $P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is conservative, then $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. If the region is **simply connected** (means **no holes**) then the converse is true, and f is both $\int P(x,y)\,dx$ and $\int Q(x,y)\,dy$.

Surfaces: If $\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|}$, **flux** is $\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS$.

As a graph: $z = f(x,y)$ and $\mathbf{N} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$ and $dS = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\,dA_{xy}$.

Parametrically: $\mathbf{P}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ and $\mathbf{N} = \frac{\partial \mathbf{P}}{\partial u} \times \frac{\partial \mathbf{P}}{\partial v}$ and $dS = \left\|\frac{\partial \mathbf{P}}{\partial u} \times \frac{\partial \mathbf{P}}{\partial v}\right\| dA_{uv}$.

If $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ and $\mathbf{F}$ is a vector field then $\begin{cases} \text{curl } F = \nabla\times\mathbf{F}, \text{ a vector field.} \\ \text{div } F = \nabla\cdot\mathbf{F}, \text{ a function.} \end{cases}$

Potentials in $\mathbb{R}^3$

If $\mathbf{F} = \nabla f$ and C is a curve, then $\int_C P\,dx + Q\,dy + R\,dz = f(\text{THE END}) - f(\text{THE START})$, path independence holds, the work over a closed curve is 0, and $\text{curl}(\nabla f) = 0$. Conversely, if F is defined in all of $\mathbb{R}^3$ with curl $F = 0$ (the cross-partials "match") then $\mathbf{F}$ has a potential, f, so $\nabla f = \mathbf{F}$. f is obtained by comparing partial integrals of the components of $\mathbf{F}$.

Stokes' Theorem

$\int\int_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,dS = \int_C \mathbf{F}\cdot d\mathbf{r} \quad \left[= \int_C P\,dx + Q\,dy + R\,dz\right]$

Divergence Theorem

$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS = \int\int\int_E \text{div } F\,dV \ \left[= \int\int\int_E \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\,dV\right]$