12. Suppose that the position of a point in \( \mathbb{R}^2 \) is given by \[ \begin{align*}
    x &= e^t \cos t \\
    y &= e^t \sin t 
\end{align*} \]
a) Carefully compute the velocity vector, \( \mathbf{v}(t) \), and the acceleration vector, \( \mathbf{a}(t) \).
b) Compute the length of the curve from \( t = 0 \) to \( t = 2\pi \).

**Comment** Your answer(s) should be exact. Answers may use traditional mathematical constants such as \( \pi \) and \( e \) and operations involving arithmetic and root extraction.
c) Compute the angle between the position vector and the acceleration vector, and show that the angle does not depend upon \( t \). What is the angle?

10. a) If \( F(x, y) = \frac{3x - 4y}{\sqrt{x^2 + y^2}} \), briefly explain why \( \lim_{(x,y) \to (0,0)} F(x, y) \) does not exist.
b) If \( G(x, y) = \frac{3x^2 - 4y^2}{\sqrt{x^2 + y^2}} \), briefly explain why \( \lim_{(x,y) \to (0,0)} G(x, y) \) exists.

10. 3. Suppose that \( f \) is a differentiable function of one variable. If \( z = f \left( \frac{xy}{x^2 + y^2} \right) \) prove that \( \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0 \).

12. 4. Find all critical points of the function \( K(x, y) = (y^2 + x) e^{(-x^2/2)} \). Describe (as well as you can) the type of each critical point. Explain your conclusions.

12. 5. Suppose \( f(x, y, z) = x^3 + y^2z \).
   a) Find an equation of the tangent plane for the level surface of \( f \) which passes through \( (2,1,-3) \).
   You do **not** need to “simplify” your answer!
   b) In what direction will \( f \) increase most rapidly at \( (2,1,-3) \)? Write a unit vector in that direction.
   You do **not** need to “simplify” your answer!
c) What is the directional derivative of \( f \) at \( (2,1,-3) \) in the direction found in b)?
   You do **not** need to “simplify” your answer!

12. 6. Suppose \( f(x, y) = \begin{cases} 
    x & \text{if } x > 0 \\
    2y & \text{if } x \leq 0 \text{ and } y > 0 \\
    0 & \text{otherwise}
\end{cases} \)
   a) For which \( (x, y) \) in \( \mathbb{R}^2 \) is \( f \) **not** continuous?
   (Just write your answer carefully. You need **not** give supporting reasons.)
   b) There is \( H > 0 \) so that if \( \|(x,y)-(0,0)\| < H \) then \( |f(x,y) - f(0,0)| < \frac{1}{1,000} \).
   Find such an \( H > 0 \) and explain why your assertion is correct.
   **Note** Any correct \( H > 0 \) is acceptable, but verification must be given.
(12) 7. A point is moving along the curve below in the direction indicated. Its motion is parameterized by arc length, \( s \), so that it is moving at unit speed. Arc length is measured from the point \( P \) (both backward and forward). The curve is intended to continue indefinitely both forward and backward in \( s \), with its forward motion curling more and more tightly around the indicated circle, \( B \), and, backward, curling more and more tightly around the other circle, \( A \). Near \( P \) the curve is parallel to the indicated line segment.

Sketch a graph of the curvature, \( \kappa \), as a function of the arc length, \( s \). What are \( \lim_{s \to +\infty} \kappa(s) \) and \( \lim_{s \to -\infty} \kappa(s) \)? Use complete English sentences to briefly explain the numbers you give.

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
\hline
K & 1 & \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{2} & 1 & \infty \\
\hline
s & -10 & -9 & -8 & -7 & -6 & -5 & -4 & -3 \\
\hline
\end{array}
\]

Note that the units on the horizontal and vertical axes differ in length.

\[
\lim_{s \to -\infty} \kappa(s) = \quad \lim_{s \to +\infty} \kappa(s) =
\]

(12) 8. The polynomial equation \( f(x, y, z) = 2xy + x^2 + 5y^3z + z^4 = 6 \) is satisfied by the point \( p = (0, -1, 2) \) (be careful of the order of the variables – check that this is correct by substituting!). Suppose now that we change the first two coordinates of \( p \) and get a point \( q = (0.03, -1.05, \ldots) \). Use linear approximation to find an approximate value for the third \( z \) coordinate of \( q \) if \( q \) also satisfies the equation \( f(x, y, z) = 6 \).

You do not need to “simplify” your answer!

(8) 9. The vector \( \mathbf{v} \) is \( 3\mathbf{i} - 7\mathbf{j} + \mathbf{k} \) and the vector \( \mathbf{w} \) is \( 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} \). Write \( \mathbf{v} \) as a sum of two vectors, \( \mathbf{v}_\parallel \) and \( \mathbf{v}_\perp \), where \( \mathbf{v}_\parallel \) is a scalar multiple of \( \mathbf{w} \) and \( \mathbf{v}_\perp \) is a vector orthogonal to \( \mathbf{w} \).
First Exam for Math 291, section 1

March 13, 2003

NAME ____________________________________________

Do all problems, in any order.

No notes or texts may be used on this exam.
You may use a calculator during the last 20 minutes of the exam.

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