1. Suppose that the position of a point in $\mathbb{R}^2$ is given by \[
\begin{cases} 
    x = e^t \cos t \\
    y = e^t \sin t
\end{cases}
\]
   a) Carefully compute the velocity vector, $\mathbf{v}(t)$, and the acceleration vector, $\mathbf{a}(t)$.
   **Answer** $\mathbf{v}(t) = (e^t \cos t - e^t \sin t)i + (e^t \sin t + e^t \cos t)j$; $\mathbf{a}(t) = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)i + (e^t \sin t + e^t \cos t - e^t \cos t)i + 2e^t \cos tj$.

   b) Compute the length of the curve from $t = 0$ to $t = 2\pi$.
   **Answer** This is \[
   \int_0^{2\pi} |\mathbf{v}(t)| \, dt = \int_0^{2\pi} \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2} \, dt = \int_0^{2\pi} \sqrt{e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t) + e^{2t}(\sin^2 t + 2\sin t \cos t + \cos^2 t)} \, dt
   \]
   \[
   = \int_0^{2\pi} \sqrt{e^{2t} + e^{2t}} \, dt = \int_0^{2\pi} \sqrt{2e^{2t}} \, dt = \sqrt{2}(e^{2\pi} - 1).
   \]

   c) Compute the angle between the position vector and the acceleration vector, and show that the angle does not depend upon $t$. What is the angle?
   **Answer** $\mathbf{r}(t) = e^t \cos ti + e^t \sin tj$ so $\mathbf{v}(t) = (e^t \cos t)(-2e^t \sin t) + (e^t \sin t)(2e^t \cos t) = 0$. Therefore $\mathbf{r}(t) \perp \mathbf{a}(t)$: they are always orthogonal. The angle is always $\frac{\pi}{2}$.

2. a) If $F(x, y) = \frac{3x - 4y}{\sqrt{x^2 + y^2}}$, briefly explain why $\lim_{(x,y)\to(0,0)} F(x,y)$ does not exist.
   **Answer** As $x \to 0^+$ with $y = 0$ (that is, along the positive $x$-axis), $F(x,y) = \frac{3x}{x} = 3$. But as $y \to 0^+$ with $x = 0$ (that is, along the positive $y$-axis), $F(x,y) = \frac{-4y}{y} = -4$. But limits should be unique, and since $3 \neq -4$, the limit does not exist.

   b) If $G(x, y) = \frac{3x^2 - 4y^2}{x^2 + y^2}$, briefly explain why $\lim_{(x,y)\to(0,0)} G(x,y)$ exists.
   **Answer** We use polar coordinates here: \[
   \begin{align*}
   x &= r \cos \theta \\
   y &= r \sin \theta
   \end{align*}
   \]
   $G(x,y) = G(r \cos \theta, r \sin \theta) = \frac{3r^2 \cos^2 \theta - 4r^2 \sin^2 \theta}{r^2} = r^2(3 \cos^2 \theta - 4 \sin^2 \theta)$.

   As $(x,y) \to (0,0)$, $r \to 0^+$. The parenthesized expression, $(3 \cos^2 \theta - 4 \sin^2 \theta)$, is bounded between $-4$ and $3$. The product of a bounded expression and a term approaching $0$ must go to $0$. So the limit exists and is $0$.

3. Suppose that $f$ is a differentiable function of one variable. If $z = f \left( \frac{xy}{x^2 + y^2} \right)$ prove that \[
   x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.
   \]
   **Answer** The Chain Rule applies: \[
   \frac{\partial}{\partial x} f \left( \frac{xy}{x^2 + y^2} \right) = f' \left( \frac{xy}{x^2 + y^2} \right) \frac{\partial}{\partial x} \left( \frac{xy}{x^2 + y^2} \right).
   \]
   But \[
   \frac{\partial}{\partial x} \left( \frac{xy}{x^2 + y^2} \right) = \frac{x(y^2) - 2xy}{(x^2 + y^2)^2} = \frac{-xy^2}{(x^2 + y^2)^2}.
   \]
   Therefore \[
   x \frac{\partial z}{\partial x} = f' \left( \frac{xy}{x^2 + y^2} \right) \frac{xy}{x^2 + y^2} \frac{\partial}{\partial x} \left( \frac{xy}{x^2 + y^2} \right).
   \]
   So one more computation is needed: \[
   \frac{\partial}{\partial y} f \left( \frac{xy}{x^2 + y^2} \right) = f' \left( \frac{xy}{x^2 + y^2} \right) \frac{\partial}{\partial y} \left( \frac{xy}{x^2 + y^2} \right) = \frac{x^2y^2 + 2xy^2}{(x^2 + y^2)^2} = \frac{-xy^2}{(x^2 + y^2)^2}.
   \]
   The second part of the partial derivative equation’s left-hand side is $y \frac{\partial z}{\partial x}$.

4. Find all critical points of the function $K(x, y) = (y^2 + x)e^{-x^2/2}$. Describe (as well as you can) the type of each critical point. Explain your conclusions.
   **Answer** Since $K_x = 1e^{-x^2/2} + (y^2 + x)e^{-x^2/2}(-x) = (1 - x^2 - xy^2)e^{-x^2/2}$ and $K_y = 2ye^{-x^2/2}$, the critical points are solutions of \[
   \left\{ \begin{array}{l}
   1 - x^2 - xy^2 = 0 \\
   2y = 0
   \end{array} \right.
   \]
   (the exponential is never 0). The c.p.’s are $A = (1,0)$ and $B = (-1,0)$. Now the Hessian: $K_{xx} = (-2xe^{-x^2/2} + (1 - x^2 - xy^2)e^{-x^2/2}(-x) = (-3x - y^2 + x^3 + x^2y^2)e^{-x^2/2}$,
   $K_{xy} = (-2xy)e^{-x^2/2}$, $K_{yx} = 2ye^{-x^2/2}(-x) = (-2xy)e^{-x^2/2}$, and $K_{yy} = 2e^{-x^2/2}$. I am happy that my $K_{xy}$ and $K_{yx}$ coincide. All four expressions have the exponential. It is always positive, so it won’t affect the sign of the Hessian. I’ll delete it in computing $H$. $H = \det \begin{pmatrix} -3x - y^2 + x^3 + x^2y^2 & -2xy \\
   -2xy & 2 \end{pmatrix}$. At $A$ and $B$, $y = 0$ and $x = \pm 1$, so $H = \det \begin{pmatrix} -3(\pm 1) \pm 1 & 0 \\
   0 & 2 \end{pmatrix} = \det \begin{pmatrix} -2(\pm 1) & 0 \\
   0 & 2 \end{pmatrix}$. Therefore, at $A$, $H = -4$ and we have a saddle point, and, at $B$, $H = 4$ and $K_{xx} > 0$, so we have a local minimum.

   **Comment** Here’s a Maple picture of the relevant part of the surface. You may be able to see the saddle and the minimum.
5. Suppose \( f(x, y, z) = x^3 + y^2 z \).
   a) Find an equation of the tangent plane for the level surface of \( f \) which passes through \((2, 1, -3)\).
   Answer \( \nabla f(x, y, z) = 3x^2 \mathbf{i} + 2y \mathbf{j} + y^2 \mathbf{k} \), so \( \nabla f(2, 1, -3) = 12\mathbf{i} - 6\mathbf{j} + 1\mathbf{k} \) and an equation is \( 12(x - 2) - 6(y - 1) + (z + 3) = 0 \).
   b) In what direction will \( f \) increase most rapidly at \((2, 1, -3)\)? Write a unit vector in that direction.
   Answer The unit vector desired is \( \frac{12\mathbf{i} - 6\mathbf{j} + \mathbf{k}}{\sqrt{12^2 + 6^2 + 1^2}} \).
   c) What is the directional derivative of \( f \) at \((2, 1, -3)\) in the direction found in b)? Answer \( \sqrt{12^2 + 6^2 + 1^2} \).

6. Suppose \( f(x, y) = \begin{cases} x & \text{if } x > 0 \\ 2y & \text{if } x \leq 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases} \)
   a) For which \((x, y)\) in \( \mathbb{R}^2 \) is \( f \) not continuous? (Just write your answer carefully. You need not give supporting reasons.)
   Answer The points to worry about are near the “edges” of the pieces: the \( x \)- and \( y \)-axes. Considering a few cases gives this answer: \( f \) is not continuous for points of the form \((0, t)\) where \( t > 0 \): the positive \( y \)-axis.
   Maple doesn’t handle the graph of surfaces with discontinuities well, so I won’t show you the result of the commands \( f:=\langle x,y\rangle\rightarrow \text{piecewise}(x>0,x,x\leq0\text{ and } y>0,2*y,0)\)
   b) There is \( H > 0 \) so that if \( ||(x, y) - (0, 0)|| < H \) then \( |f(x, y) - f(0, 0)| < \frac{1}{1000} \). Find such \( H > 0 \) and explain why your assertion is correct. Note: Any correct \( H > 0 \) is acceptable, but verification must be given.
   Answer \( H = \frac{1}{1000} \) will work. For if \( ||(x, y)|| < \frac{1}{1000} \), \( x \) and \( y \) will both be less than \( \frac{1}{1000} \), and each of the non-zero pieces of the definition \((x, 2y)\) will surely be less than \( \frac{1}{1000} \). Since \( f(0, 0) = 0 \), this guarantees that the difference \( f(x, y) - f(0, 0) \) must be less than \( \frac{1}{1000} \). Many \( H \)’s are good answers to this question.

7. A point is moving along the curve below in the direction indicated. Its motion is parameterized by arc length, \( s \), so that it is moving at unit speed.
   Arc length is measured from the point \( P \) (both backward and forward). The curve is intended to continue indefinitely both forward and backward in \( s \), with its forward motion curling more and more tightly around the indicated circle, \( B \), and, backward, curling more and more tightly around the other circle, \( A \). Near \( P \) the curve is parallel to the indicated line segment.
   Sketch a graph of the curvature, \( \kappa \), as a function of the arc length, \( s \). What are \( \lim_{s \to +\infty} \kappa(s) \) and \( \lim_{s \to -\infty} \kappa(s) \)?
   Use complete English sentences to briefly explain the numbers you give.
   Answer I hope to see these features: as \( s \to -\infty \), \( \kappa \to 1 \): the curve is getting closer to a circle of radius 1 with curvature \( = 1 \). There should be an interval of length \( \approx 4 \) centered at \( s = 0 \) where the curve is “flat”\( \kappa = 0 \).
   As \( s \to +\infty \), \( \kappa \to \frac{1}{2} \): the curve is getting closer to a circle of radius 2 with curvature \( = \frac{1}{2} \). The graph should decrease and then increase. Also, \( \lim_{s \to +\infty} \kappa(s) = \frac{1}{2} \).

8. The polynomial equation \( f(x, y, z) = 2xy + x^2 + 5y^2z + z^4 = 6 \) is satisfied by the point \( p = (0, -1, 2) \)
   (be careful of the order of the variables – check that this is correct by substituting!). Suppose now that we change the first two coordinates of \( p \) and get a point \( q = (0.3, -1.05, \ldots) \). Use linear approximation to find an approximate value for the third \( (z) \) coordinate of \( q \) if \( q \) also satisfies the equation \( f(x, y, z) = 6 \).
   Answer Since \( z \) is the independent variable, we will differentiate \( f = 6 \) by \( x \) and \( y \) respectively to get the relevant partial derivatives. \( \frac{\partial f}{\partial x} = 2y + 2x + 10yz \frac{\partial z}{\partial x} + 4z^3 \frac{\partial z}{\partial x} = 0 \).
   At \( p = (0, -1, 2) \) this becomes \( -2 + 5(-1)^3 \frac{\partial z}{\partial x} + 32 \frac{\partial z}{\partial x} = 0 \) so that \( \frac{\partial z}{\partial x} = \frac{2}{27} \).
   Therefore \( \Delta z \approx \frac{\partial x}{\partial x} \Delta x + \frac{\partial z}{\partial x} \Delta y = \frac{2}{27}(0.3) - \frac{2}{27}(-0.5) \) and the approximate value of \( z \) is \( 2 \) plus all that.
   Comment: Maple tells me that \( 2 + \frac{2}{27}(0.3) - \frac{2}{27}(-0.5) \approx 2.05778 \). \( \text{solve} \) gives 2.05926, quite close. Please use \( \text{implicitplot3d} \) to examine \( f = 6 \) when \( -5 \leq x, y, z \leq 5 \): you’ll see a lovely surface with a hole in it!

9. The vector \( \mathbf{v} = 3\mathbf{i} - 7\mathbf{j} + \mathbf{k} \) and the vector \( \mathbf{w} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k} \). Write \( \mathbf{v} \) as a sum of two vectors, \( \mathbf{v}_\parallel \) and \( \mathbf{v}_\perp \), where \( \mathbf{v}_\parallel \) is a scalar multiple of \( \mathbf{w} \) and \( \mathbf{v}_\perp \) is a vector orthogonal to \( \mathbf{w} \).
   Answer \( \mathbf{v}_\parallel = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||^2} \mathbf{w} \), and \( \mathbf{v} \cdot \mathbf{w} = 3 \cdot 2 - 7 \cdot 1 + 1 \cdot (3) = -4 \) and \( ||\mathbf{w}||^2 = 2^2 + 1^2 + (-3)^2 = 14 \).
   Therefore \( \mathbf{v}_\parallel = -\frac{1}{14}(3\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = -\frac{3}{14}\mathbf{i} - \frac{1}{14}\mathbf{j} + \frac{3}{14}\mathbf{k} \).
   \( \mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel = 3\mathbf{i} - 7\mathbf{j} + \mathbf{k} - (-\frac{3}{14}\mathbf{i} - \frac{1}{14}\mathbf{j} + \frac{3}{14}\mathbf{k}) = \frac{50}{14}\mathbf{i} - \frac{9}{14}\mathbf{j} + \frac{13}{14}\mathbf{k} \).
   Now \( \mathbf{v}_\perp \cdot \mathbf{w} = \frac{200 + 1 \cdot (-9) + 2 \cdot (-3) \cdot 2}{14} = 0 \) so maybe this answer is correct.