Math 291:01  Solution of the second exam’s bonus problem  11/28/2002

Prove Green’s Theorem \((\int_C P(x, y) \, dx + Q(x, y) \, dy = \int \int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA)\) if \(R\) is the triangular region in the \(xy\)-plane which is bounded by the positive \(x\)-axis, the positive \(y\)-axis, and line 
\[x + y = 1,\] and if \(C\) is the oriented boundary of \(R\): the line segment from \((0,0)\) to \((1,0)\), followed by the line segment from \((1,0)\) to \((0,1)\), and completed by the line segment from 
\((0,1)\) to \((0,0)\). \(P(x,y)\) and \(Q(x,y)\) are functions with continuous first partial derivatives on \(C\) and all of \(R\).

Answer Begin with \(\int \int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA\). This is \(\int \int_R \frac{\partial Q}{\partial x} \, dA - \int \int_R \frac{\partial P}{\partial y} \, dA\). Now convert to iterated integrals, with a careful choice of order:
\[
\int_0^1 \int_0^{1-x} \frac{\partial Q}{\partial x} \, dx \, dy - \int_0^1 \int_0^{1-y} \frac{\partial P}{\partial y} \, dy \, dx \cdot
\]
Apply the Fundamental Theorem of Calculus in the first integral with respect to \(x\) and in the second integral with respect to \(y\). We get a sum of four integrals:
\[
\int_0^1 (Q(y,1-y) - Q(0,y)) \, dx - \int_0^1 (P(x,1-x) - P(x,0)) \, dx = \int_0^1 Q(y,1-y) \, dy - \int_0^1 Q((0,y)) \, dy - \int_0^1 P(x,1-x) \, dx + \int_0^1 P(x,0) \, dx.
\]

Now consider \(\int_C P(x, y) \, dx + Q(x, y) \, dy\). Write \(C = C_1 + C_2 + C_3\) where \(C_1\) is the line segment from \((0,0)\) to \((1,0)\), \(C_2\) is the line segment from \((1,0)\) to \((0,1)\), and \(C_3\) is the line segment from \((0,1)\) to \((0,0)\).

The \(C_1\) integral First parameterize the line segment from \((0,0)\) to \((1,0)\):
\[
\begin{cases}
0 \leq t \leq 1. \text{ Then } \int_C P(x, y) \, dx + Q(x, y) \, dy = \int_0^1 P(t,0) \, dt.
\end{cases}
\]

The \(C_3\) integral Parameterize the line segment from \((1,0)\) to \((0,0)\):
\[
\begin{cases}
0 \leq t \leq 1 \text{ but with a minus sign, since the direction is “down”}\. \text{ Then } \int_C P(x, y) \, dx + Q(x, y) \, dy = -\int_0^1 Q(0,t) \, dt.
\end{cases}
\]

The \(C_2\) integral This is the most interesting case. Begin with \(\begin{cases} x = t \, y = 1-t \end{cases}\) so that
\[
\begin{cases}
0 \leq t \leq 1, \text{ again prefixing the result with a minus sign because of the direction}. \text{ Then } \int_C P(x, y) \, dx + Q(x, y) \, dy = -\int_0^1 (P(t,1-t) - Q(t,1-t)) \, dt. \text{ The value of this is the same as the sum of the integral } A \text{ and the integral } C, \text{ with } C’s \text{ minus sign included.}
\end{cases}
\]

We have verified the equality in Green’s Theorem for this triangle.