1. a) Find an equation for the plane tangent to the surface \( x^2 yz = 6 \) at the point \((1,2,3)\).

**Answer** The gradient is \(2xyz + x^2 j + x^2 yk\) which at \((1,2,3)\) is \(12j + 3j + 2k\). So we have a normal vector and a point on the plane, and an equation for the plane is \(12(x - 1) + 3(y - 2) + 2(z - 3) = 0\).

b) A particle has position vector given by \(\mathbf{R}(t) = \cos(2t)i + \sin(t^2)j - 3k\). Find parametric equations for a line tangent to the path of this particle when \(t = 0\).

**Answer** The velocity vector is \(-2\sin(2t)i + \cos(t^2)2j - 3k\) which at \(t = 0\) is \(0i + 0j - 3k\). Also note that \(\mathbf{R}(0) = 1i + 0j + 0k\). Now we have a point on the line and a vector in the direction of the line. So parametric equations for the line are \(x = 0t + 1\) and \(y = 0t + 0\) and \(z = -3t + 0\).

c) The plane found in a) and the line found in b) intersect. Find the point of intersection.

**Answer** Put the answer to b) in the answer to a) and solve. Get \(t\) and then get the point: \(12((0t + 1) - 1) + 3((0t + 0) - 2) + 2((-3t + 0) - 3) = 0\). This gives us \(t = -2\) so the point we are looking for is \((1,0,6)\).

Cheap check: see that \((1,0,6)\) also satisfies the equation for the plane, which it does.

2. Suppose \(\begin{cases} x = u + v^2 \\ y = 2uv - v^3 \end{cases}\). Note that when \(u = 3\) and \(v = 1\), then \(x = 4\) and \(y = 5\). a) Suppose \(u\) is changed to \(3.02\) and \(v\) is changed to \(0.96\). Use linear approximations to estimate the “new” values of \(x\) and \(y\).

**Answer** \(x_u = 1\) and \(x_v = 2v\) and \(y_u = 2u - 3v^2\), so at \(u = 3\) and \(v = 1\), \(x_u = 1\) and \(x_v = 2\) and \(y_u = 2\) and \(y_v = 3\). Also, \(\Delta u = +0.02\) and \(\Delta v = -0.04\). Linearization gives these equations:
\[
\begin{align*}
\Delta x &\approx x_u \Delta u + x_v \Delta v = 1 + 0.02 + 2(-0.04) = -0.06 \\
\Delta y &\approx y_u \Delta u + y_v \Delta v = 2 + 0.02 + 3(-0.04) = -0.08
\end{align*}
\]
Thus the “new” (approximate) values of \(x\) and \(y\) are \(3.94\) and \(4.92\). **Comment** The “true” new values (exactly computed) are \(3.9416\) and \(4.91664\).

b) Suppose that we wish to estimate what values of \(u\) and \(v\) near \(u = 3\) and \(v = 1\) will give the values \(x = 4.07\) and \(y = 5.03\). Use linear approximations backwards to estimate such values.

**Answer** \(\Delta x\) and \(\Delta y\) are known (one \(0.07\) and the other \(0.03\)). We want to estimate \(\Delta u\) and \(\Delta v\). They are related by the approximate equations:
\[
\begin{align*}
\Delta x &\approx 1\Delta u + 2\Delta v \\
\Delta y &\approx 3\Delta u - 3\Delta v
\end{align*}
\]
Treat this as a system of two linear equations in two unknowns. Doubling the first and subtracting it from the second gives us: \(-11 \approx -\Delta v\), so \(\Delta v \approx 11.11\). “Plugging” this value back into the first equation gets \(\Delta u \approx 0.07 - 2 \cdot 0.07 = -0.1.11\). You can check this, as I did, by substituting the values back into the original linear equations. Thus the new values of \(u\) and \(v\) are \(\begin{cases} \text{new } u = u + \Delta u \approx 3 + (-1.11) = 2.89 \\ \text{new } v = v + \Delta v \approx 1 + 1.11 = 2.11 \end{cases}\) **Comment** If we substitute in the obtained new values for \(u\) and \(v\) we get \(4.0821\) for \(x\) and \(4.959369\) for \(y\). I can’t explain the (relatively) large discrepancy in \(y\).

3. Suppose \(f(x,y) = x^2 y\). Then \(f(2,3) = 12\). There is \(H > 0\) so that if \(||(x,y) - (2,3)|| < H \) then \(|f(x,y) - f(2,3)| < \frac{1}{1000}\). Find such an \(H\) and explain why your assertion is correct.

**Answer** \(|x^2 y - 2^2 \cdot 3| = |(x^2 y - 2^2 y) + (2^2 y - 2^2 \cdot 3)| \leq |x^2 y - 2^2 y| + |2^2 y - 2^2 \cdot 3|\). I’ll do the first part:
\[|x^2 y - 2^2 y| = |y||x - 2| + 2|y||x - 2|\] We also know that \(\text{IF}\ |y - 3| < 1\) then \(|y| < 3\). We may then (over-)estimate \(|x^2 y - 2^2 y|\) by \(4 \cdot 3|y| = 12|y|\). \(\text{IF}\ |x - 1| < \frac{1}{1000}\) and if the other restrictions (the other \(2\text{IF}\’s\)) are fulfilled, this part will be less than \(\frac{1}{1000}\). For the second part: \(|2^2 y - 2^2 \cdot 3| \leq 4|y| - 3|\). This will be less than \(\frac{1}{1000}\). \|\text{IF}\ |y - 3| < \frac{1}{1000}\|. So if we take \(H = \frac{1}{1000}\), then all four \(\text{IF}\’s\) are satisfied, and each of the two parts is less than \(\frac{1}{1000}\), so that \(|f(x,y) - f(1,2)| < \frac{1}{2000} + \frac{1}{2000} < \frac{1}{1000}\). Of course this is not the only valid answer for \(H\).

4. If \(x = s^2 - t^2\), \(y = 2st\), and \(z = F(x,y)\), show that \(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 4\sqrt{x^2 + y^2} \left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right)\).

**Answer** I’ll use subscripts instead of \(\partial\’s\) here. The Chain Rule says that \(z_x = z_x x_s + z_y y_y = z_x 2s + z_y 2t\). Differentiate this again with respect to \(s\), remembering the Chain Rule (both \(z_x\) and \(z_y\) may be functions of \(s\) and \(t\)) and using the product rule: \(z_{xx} = (z_{xx})_s + z_{xx} 2s + z_{yy} y_s + (z_y)_s 2t + z_y 2t\) = \((z_{xx})_s + z_{xx} y_s + z_{yy} 2s + 2z_x + (z_{xx})_s + z_{yy} 2t + z_y (0)) = (4s^2)z_{xx} + (8st) |z_{xx}| st + (4t^2)z_{yy} + 2z_x\). Now I differentiate once more with respect to \(t\): \(z_{tt} (z_{xx})_s + z_{xx} 2s + z_{yy} y_s + (z_y)_s 2t + z_y 2t\) = \((z_{xx})_s + z_{xx} y_s + z_{yy} 2s + (z_{xx})_t 2s - 2z_x + z_{yy} 2t\) = \((4t^2)z_{xx} - (8st)z_{xx} + (4s^2)z_{yy} - 2z_x\). The sum of the two underlined expressions, after cancellation and
factoring, is $z_{xx} + z_{tt} = 4(s^2 + t^2)(z_{xx} + z_{yy})$. Since $x = s^2 - x^2$ and $y = 2st, x^2 + y^2 = s^2 - 2st^2 + t^4 + 4(st)^2 = s^4 + 2s^2 t^2 + t^4 = (s^2 + t^2)^2$, and $\sqrt{x^2 + y^2} = s^2 + t^2$. So we’ve verified the requested formula.

(12) 5. Suppose $F(a, b, c) = a^2 b + \sqrt{bc + 2c}$. a) Compute $\frac{\partial F}{\partial a}, \frac{\partial F}{\partial b}$, and $\frac{\partial F}{\partial c}$ at the point $p$ with coordinates $a = 2, b = 1$, and $c = 3$.

Answer $\nabla F(a, b, c) = 2ab \mathbf{j} + \left(\frac{1}{2\sqrt{bc + 2c}}\right) \mathbf{k}$. At $(2, 1, 3)$ we get $4i + 4\mathbf{j} + 2\mathbf{k}$. At $(2, 1, 3)$ we get $4i + 4\mathbf{j} + 2\mathbf{k}$. b) In what direction will $F$ increase most rapidly at $p$? Write a unit vector in that direction.

Answer $\frac{4i + 4\mathbf{j} + \mathbf{k}}{\sqrt{4^2 + 4^2 + 2^2}}$.

c) What is the directional derivative of $F$ at $p$ in the direction found in b)?

Answer $\frac{4(2) + 4(1) + (1)}{2}$.  

(12) 6. Euler investigated the following specific example: $V = x^3 + y^2 - 3xy + \frac{3}{2} x$. He asserted that $V$ has a minimum both at $x = \frac{1}{2}$ and $y = \frac{1}{2}$ and at $x = \frac{1}{2}$ and $y = \frac{3}{2}$. Was Euler correct?

Answer $V_x = 3x^2 - 3y + \frac{3}{2}$ and $V_y = 2y - 3x$. Critical points occur where both $V_x$ and $V_y$ are 0. There $2y = 3x$ or $y = \frac{3}{2} x$, so the $V_y$ condition becomes $3x^2 - \frac{3}{2} x^2 + \frac{3}{2} = 0$ or $6x^2 - 9x = 3$ = 0 which factors* into $(2x - 1)(3x - 3) = 0$. The critical points are as Euler wrote: $(1, \frac{3}{2})$ and $(\frac{3}{2}, \frac{1}{2})$. Now test the type of the critical points: $V_{xx} = 6x, V_{xy} = -3, V_{ux} = -3, V_{uy} = 2$. So the Hessian is det $\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix} = 12x - 9$. At $(1, \frac{3}{2})$, the Hessian is $12 - 9 > 0$, and $V_{xx} = 6 > 0$, so this critical point is a local minimum. At $(\frac{3}{2}, \frac{1}{2})$, the Hessian is $12 - 9 = 3 < 0$, which makes this critical point a saddle point. Euler was wrong!

(10) 7. A particle has position vector given by $\mathbf{R}(t) = t^\frac{1}{2}i + t^2j - 3tk$.

a) What are the velocity and acceleration vectors of this particle when $t = 1$?

Answer $\mathbf{v}(t) = -\frac{1}{2}i + 2\mathbf{j} - 3\mathbf{k}$ so $\mathbf{v}(1) = -i + 2\mathbf{j} - 3\mathbf{k}$. Also, $\mathbf{a}(t) = \frac{1}{3}i + 2\mathbf{j} + 0\mathbf{k}$ so $\mathbf{a}(1) = 2\mathbf{i} + 2\mathbf{j}$.

b) Write the acceleration vector when $t = 1$ as a sum of two vectors, one parallel to the velocity vector when $t = 1$ and one perpendicular to the velocity vector when $t = 1$.

Answer $|\mathbf{v}(1)| = \sqrt{1 + 4 + 9} = \sqrt{14}$, and $\mathbf{a}(1) \cdot \mathbf{v}(1) = -2 + 4 = 2$ so that $\mathbf{a} = \frac{\mathbf{a}(1) \cdot \mathbf{v}(1)}{|\mathbf{v}(1)|^2} \mathbf{v}(1) = \frac{2}{\sqrt{14}}(-i + 2\mathbf{j} - 3\mathbf{k})$. Normal component: $\mathbf{a} = \mathbf{v} - \mathbf{a} \cdot \mathbf{v}(1) = \left(\frac{1}{2}i + \frac{2}{3}j + \frac{2}{3}k\right) \cdot (-i + 2\mathbf{j} - 3\mathbf{k}) = -\frac{1}{2} + \frac{4}{3} - \frac{6}{3} = 0$ so that the “normal” component is perpendicular to the velocity vector, as it’s supposed to be.

(8) 8. The curve below is parameterized by arc length, $s$. Arc length is measured forward and backward from the indicated initial point where $s = 0$. Sketch a graph of the curvature, $\kappa(s)$, of this curve as well as you can.

Answer There’s not much quantitative information given in the initial graph, so the “qualitative” information must be checked. Such a response should show some symmetry around $s \approx 2.2$. I thought it should be “unimodal”: increasing before the axis of symmetry, and decreasing after. Some students sketched a graph with “shoulders”. I experimented with Maple and some models show that such a sketch is possible and even reasonable! At the edges, the curvature should be small. No supporting discussion was requested.

(6) 9. Explain briefly why the following limit does not exist. $\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2 + y^2}$

Answer We look at $(x, y) \neq (0,0)$ here. If $y = x^2$, then $\frac{x^2 y}{x^2 + y^2} = \frac{x^6}{2x^4} = 2$. So the limit along the parabolic path $y = x^2$ as $(x, y) \to (0,0)$ is 2. But along either axis (with $x = 0$ or $y = 0$), $\frac{x^2 y}{x^2 + y^2} = 0$. Since $0 \neq 2$, the limit does not exist.

* Even then textbook problems were predictable.