640:151:04–06 **Answers to version B of the second exam** **11/12/2003**

(8) 1. Suppose that $f(x) = \sqrt{17 - x^3}$. Then $f(2) = 3$.

a) Use linear approximation to get an approximate value for $f(1.97)$. You do **not** need to "simplify" your answer!

Answer $f'(x) = \frac{1}{2}(17 - x^3)^{-1/2}(-3x^2)$ so $f'(2) = -\frac{12}{6} = -2$, and $f(1.97) \approx 3 + (-.03)(-2) = 3.06$.

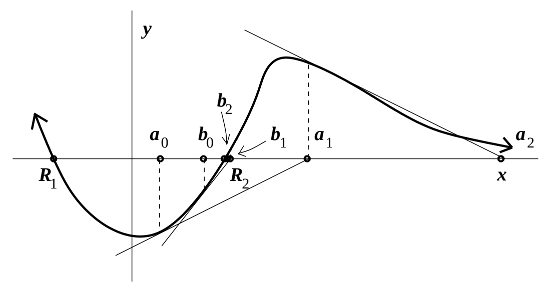
b) 3.058533472 is the true value of $f(1.97)$ to ten-digit accuracy. Explain briefly using calculus (not calculator evidence!) how you could have predicted that the true value is likely to be greater than or less than the approximate value found in a).

Answer $f''(x) = -\frac{1}{4}(17-x^3)^{-3/2}(-3x^2)^2 + \frac{1}{2}(17-x^3)^{-1/2}(-6x)$ so $f''(2) = -\frac{1}{4}\cdot\frac{1}{27}(-12)^2 + \frac{1}{2}\cdot\frac{1}{3}(-12)$, which is negative. The curve is concave down near $x = 2$, and most likely the linear approximation overestimates the true value. This is actually correct.

(8) 2. Below is a graph of the function $y = f(x)$. The $x-$axis is a horizontal asymptote of this graph as $x \to \infty$ and the function f steadily decreases for large x. As $x \to -\infty$ the values of f get larger without any upper bound. Newton's method will be used to approximate a root of $f(x) = 0$.

a) If Newton's method is applied repeatedly with initial value $x = a_0$, a sequence of numbers $\{a_n\}$ is obtained (here a_{n+1} is the result of applying Newton's method to a_n). Draw a_1 and a_2 on the graph above as well as you can. What will happen to a_n as $n \to \infty$?

Answer a_1 and a_2 are shown on the graph to the right. I believe $\lim_{n\to\infty} a_n = \infty$.

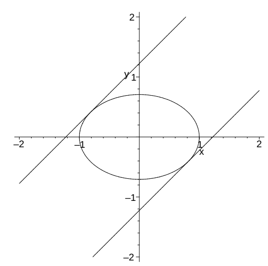


b) If Newton's method is applied repeatedly with initial value $x = b_0$, a sequence of numbers $\{b_n\}$ is obtained (here b_{n+1} is the result of applying Newton's method to b_n). Draw b_1 and b_2 on the graph above as well as you can. What will happen to b_n as $n \to \infty$?

Answer b_1 and b_2 are shown on the graph to the right. I believe $\lim_{n\to\infty} b_n = R_2$.

(10) 3. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

Answer We will $\frac{d}{dx}$ the equation $x^2 + 2y^2 = 1$ and the result is $2x + 4y\frac{dy}{dx} = 0$ so that $\frac{dy}{dx} = -\frac{x}{2y}$. We must find points (x, y) which satisfy $-\frac{x}{2y} = 1$ and $x^2 + 2y^2 = 1$. The first equation gives $x = -2y$ so the second equation becomes $4y^2 + 2y^2 = 1$ and y is $\pm\frac{1}{\sqrt{6}}$. The requested points are $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$. The accompanying picture drawn by `Maple` displays the ellipse along with lines having slope 1 tangent to the ellipse at those points.



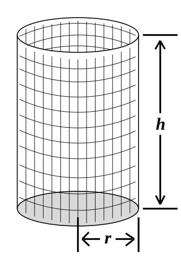
(10) 4. Find $f(x)$ if $f''(x) = 2e^x + 3\sin x$ and $f(0) = 0$ and $f'(\pi) = 0$.

Answer $f'(x) = 2e^x - 3\cos x + C$ so that $f'(\pi) = 0$ gives $2e^\pi - 3\cos\pi + C = 0$ which is $2e^\pi - 3(-1) + C = 0$ so that $C = -2e^\pi - 3$. Since $f'(x) = 2e^x - 3\cos x - 2e^\pi - 3$ we know that $f(x) = 2e^x - 3\sin x + (-2e^\pi - 3)x + C$ and $f(0) = 0$ gives $2e^0 - 3\sin 0 + 0 + C = 0$ so that $C = -2$. Therefore $f(x) = 2e^x - 3\sin x + (-2e^\pi - 3)x - 2$.

(15) 5. A cylindrical can without a top is made to contain V cm^3 of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

Note Be sure you tell why you have found a *minimum.*

Answer The metal has area $A = 2\pi rh + \pi r^2$ where h is the height of the cylinder and b is the radius of the circular base. The constraint is that $V = \pi r^2 h$ so $h = \frac{V}{\pi r^2}$. Thus we need to minimize $A(r) = \frac{2V}{r} + \pi r^2$. As $r \to \infty$ or as $r \to 0^+$, $A(r) \to \infty$, so the minimum occurs at a finite critical point. We see that $A'(r) = -\frac{2V}{r^2} + 2\pi r$. This is 0 when $r = \left(\frac{V}{\pi}\right)^{1/3}$. Then $h = \frac{V}{\pi r^2} = \frac{V}{\pi}\cdot\left(\frac{\pi}{V}\right)^{2/3} = \left(\frac{V}{\pi}\right)^{1/3}$.



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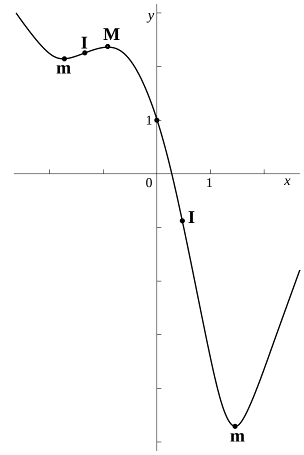
(16) 6. Suppose f is a differentiable function with $f'(x) = (x+1)(x^2-3)$. This is a formula for the *derivative* of f. In the answers to parts a)–c) give exact values, not approximations, using any traditional constants.
a) On what intervals is f increasing and on what intervals is f decreasing?
Answer The roots of $f(x) = 0$ are -1 and $\pm\sqrt{3}$. f' changes sign at each of these roots. If x is large negative, $f'(x)$ is negative, and if x is large positive, $f'(x)$ is positive. Therefore I know that f is decreasing on $(-\infty, -\sqrt{3})$ and $(-1, \sqrt{3})$ and f is increasing on $(-\sqrt{3}, -1)$ and $(\sqrt{3}, \infty)$.
b) For what values of x does f have a local maximum? For what values of x does f have a local minimum?
Answer Using the information above, f has a local maximum at $x = -1$. f has a local minimum at $x = -\sqrt{3}$ and $x = \sqrt{3}$.

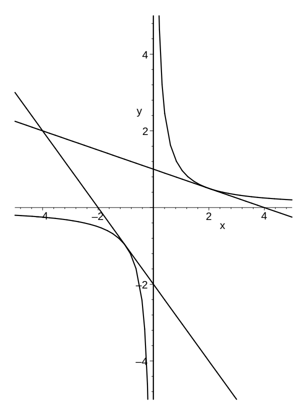


c) Compute $f''(x)$. On what intervals is f concave up and on what intervals is f concave down? **Answer** $f''(x) = 1(x^2-3) + (x+1)(2x) = 3x^2 + 2x - 3$. The roots of $3x^2 + 2x - 3 = 0$ are $\frac{-2\pm\sqrt{2^2-4(3)(-3)}}{6} = -\frac{1}{3} \pm \frac{\sqrt{40}}{6} = -\frac{1}{3} \pm \frac{\sqrt{10}}{3}$. The sign of $f''(x)$ changes at each of these points. When $|x|$ is large, $f''(x) > 0$. Therefore: $f''(x) > 0$ on $\left(-\infty, -\frac{1}{3} - \frac{\sqrt{10}}{3}\right)$ and on $\left(-\frac{1}{3} + \frac{\sqrt{10}}{3}, \infty\right)$ and f is concave up on those intervals. $f''(x) < 0$ on $\left(-\frac{1}{3} - \frac{\sqrt{10}}{3}, -\frac{1}{3} + \frac{\sqrt{10}}{3}\right)$ and f is concave down on that interval.
d) Use the information given in parts a)–c) together with the fact that $f(0) = 1$ to sketch a graph of $y = f(x)$ on the axes given. Label any inflection points with **I**, label any local maxima with **M**, and label any local minima with **m**. **Answer** An answer is shown to the right which I drew. I will look for qualitative aspects in student solutions. `Maple` helped me get the solution algebraically: $f(x)$ actually is $\frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{3}{2}x^2 - 3x + 1$.

(10) 7. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2? **Answer** $A = \frac{1}{2}bh$ so $\frac{dA}{dt} = \frac{1}{2}\frac{db}{dt}h + \frac{1}{2}b\frac{dh}{dt}$. When the altitude is 10 and the area is 100, the base must be 20. Therefore $2 = \frac{1}{2}\frac{db}{dt}10 + \frac{1}{2}20(1)$ so that $\frac{db}{dt} = -1.6$ cm/min.

(15) 8. Find the limit, which could be a specific real number or $+\infty$ or $-\infty$ or NOT EXIST. In each case, briefly indicate your reasoning, based on algebra or properties of functions or techniques of the course.
a) $\lim_{x\to\infty} x^3e^{-x}$ **Answer** $\lim_{x\to\infty} x^3e^{-x} = \lim_{x\to\infty} \frac{x^3}{e^x}$. This is an Indeterminate Form of the type $\frac{\infty}{\infty}$ and we will analyze it using l'Hopital's Rule. So: $\lim_{x\to\infty} \frac{x^3}{e^x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{3x^2}{e^x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{6x}{e^x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{6}{e^x} = 0$
b) $\lim_{x\to 0} \frac{1-\cos x}{x^2+x}$ **Answer** This is an Indeterminate Form of the type $\frac{0}{0}$ and we will analyze it using l'Hopital's Rule. So: $\lim_{x\to 0} \frac{1-\cos x}{x^2+x} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\sin x}{2x+1}$. But the second quotient is continuous at $x = 0$ so that the value of the limit is the value when $x = 0$ is "plugged in": 0.
c) $\lim_{x\to\infty} \left(\arctan(x^2) - \arctan(x)\right)$ **Answer** When A is a large positive number, $\arctan A$ is close to $\frac{\pi}{2}$. When x is large, x^2 will also be large and both $\arctan(x^2)$ and $\arctan(x)$ will be close to $\frac{\pi}{2}$. The limit exists and its value is $\frac{\pi}{2} - \frac{\pi}{2} = 0$.

(8) 9. Find all lines tangent to $y = \frac{1}{x}$ which pass through the point $(-4, 2)$.
Answer Since $y' = -\frac{1}{x^2}$, and the slope of a line connecting $(x, y) = (x, \frac{1}{x})$ with $(-4, 2)$ is $\frac{\frac{1}{x}-2}{x-(-4)}$, we know that $-\frac{1}{x^2}$ should equal $\frac{\frac{1}{x}-2}{x-(-4)}$. If we cross-multiply, we get the equation $-(x-(-4)) = x^2\left(\frac{1}{x} - 2\right)$, and this becomes $-x-4 = x-2x^2$ so that we need to solve $2x^2 - 2x - 4 = 0$ or $x^2 - x - 2 = 0$. Amazingly (or not, since it *is* a problem on an exam!) the left-hand side factors into $(x+1)(x-2)$ so the roots of the equation are -1 and 2. When $x = -1$, the point on $y = \frac{1}{x}$ is $(-1, -1)$ and the slope is -1, so that the tangent line is $(y+1) = (-1)(x+1)$. When $x = 2$, the point on $y = \frac{1}{x}$ is $(2, \frac{1}{2})$ and the slope is $-\frac{1}{4}$, so that the tangent line is $(y - \frac{1}{2}) = (-\frac{1}{4})(x-2)$. To the right is a picture of the two lines and the curve, a hyperbola, drawn by `Maple`.



References Problem 3: chap. 3 review, #64; problem 4: sect. 4.10, #40; problem 5: sect. 4.7, # 33 (the text does not have a picture); problem 7: sect. 3.10, #7; problem 8a: chap. 4 review, #8; problem 8b: chap. 4 review, #8; problem 9: exam 1, #8.