(12) 1. a) State the formal definition of the derivative, \( f'(x) \), of the function \( f(x) \).

**Answer** \( f'(x) = \lim_{{h \to 0}} \frac{{f(x+h) - f(x)}}{h} \).

b) Use your answer to a) combined with algebraic manipulation and standard properties of limits to compute the derivative of \( f(x) = \sqrt{5x+3} \).

**Answer** If \( f(x) = \sqrt{5x+3} \), then \( f(x+h) = \sqrt{5(x+h) + 3} \), and \( \frac{{f(x+h) - f(x)}}{h} = \frac{{\sqrt{5x+3} - \sqrt{5x+3+h}}}{{h}} \). Multiply top and bottom by \( (\sqrt{5x+3} + \sqrt{5x+3+h}) \) to get \( \frac{{5}}{{h(\sqrt{5x+3} + \sqrt{5x+3+h})}} \). As \( h \to 0 \), \( \sqrt{5x+3+h} - \sqrt{5x+3} = \frac{5h}{{2\sqrt{5x+3}}} \) and the limit of the difference quotient is \( \frac{5}{{2\sqrt{5x+3}}} \), which is \( f'(x) \).

(10) 2. Note that \( x^2(x-2) = x^3 - 2x^2 \).

a) Find an equation for the line tangent to \( y = x^3 - 2x^2 \) when \( x = 1 \).

**Answer** \( y' = 3x^2 - 4x \), so when \( x = 1 \), \( y' = -1 \). Also, when \( x = -1 \), \( y = -1 \).

b) Sketch the line found in a) and the curve \( y = x^3 - 2x^2 \) on the axes given below as well as you can. The units on the vertical and horizontal axes are different. **Answer** To the right.

c) For which \( x \)'s are the tangent lines to the curve \( y = x^3 - 2x^2 \) horizontal?

**Answer** \( 3x^2 - 4x = 0 \) when \( x = 0 \) and \( x = \frac{4}{3} \).

(20) 3. Find the limit, which could be a specific real number or \( +\infty \) or \( -\infty \). In each case, briefly indicate your reasoning, based on algebra or properties of functions.

a) \( \lim_{{x \to \infty}} \frac{{x^2 + 2}}{x - 2} \) **Answer** \( \frac{x^2 + 2}{x^2 - 2} = \frac{1}{1} \). As \( x \to 2 \), this \( \to \frac{1}{2} \).

b) \( \lim_{{x \to 4}} \frac{1}{|x-4|} \) **Answer** If \( x > 4 \), then \( 0 < 4 - x \) so \( |4 - x| = -(4 - x) \). Therefore \( \frac{1}{|4-x|} = \frac{1}{4-x} = -1 \). So the limit is \( -1 \).

c) \( \lim_{{x \to 10^-}} \frac{1}{x-10} - \frac{1}{x} \) **Answer** If \( x < 10 \) and close to 10, then \( x^2 < 100 \) and close to 100. So \( 100 - x^2 \) is a small positive number, and \( \frac{1}{100-x} - \frac{1}{x} = \frac{1}{100-x} - \frac{1}{x} \) will be a large positive number. The limit is \( +\infty \).

d) \( \lim_{{x \to \infty}} \frac{1}{x^2 + 2x - 2} \) **Answer** As \( x \to +\infty \), \( e^x \) grows unboundedly and \( e^{-x} \) decays to 0. Therefore \( \frac{1}{x^2 + 2x - 2} \to 0 \).

(10) 4. Suppose \( f(x) = x^2 - \frac{1}{x^2 + 12} + \sin(70x) \).

a) There is at least one number \( x \) between 0 and 2 for which \( f(x) = 0 \). Explain why this is true using complete English sentences together with appropriate references to results of this course.

**Answer** \( f(0) = 0 - \frac{1}{12} + \sin 0 = -\frac{1}{12} < 0 \). \( f(2) = 4 - \frac{1}{144} + \sin(140) \). Since \( \sin(140) \geq -1 \), we see that \( f(2) \) must be greater than \( 2 \cdot \frac{1}{12} \), which is a positive number. Now \( f \) is a continuous function since rational functions in their domain and sine are continuous. Since \( f(0) < 0 < f(2) \), the Intermediate Value Theorem implies that \( f(x) = 0 \) for at least one \( x \) in the open interval \( 0 < x < 2 \).

b) If \( x \geq 2 \), \( f(x) \) must be positive. Again, explain why this is true.

**Answer** If \( x \geq 2 \), \( f(x) \) is at least \( 2^2 - \frac{1}{14} = 4 \). This is because the largest negative sine can be is \(-1\), and when \( x > 0 \), \( \frac{1}{x^2 + 12} \) is at most \( \frac{1}{12} \) and \( x^2 \) is at least \( 2^2 \). But, as above, \( 2^2 - \frac{1}{12} = 4 \) is positive, so \( f(x) \) must be positive for \( x > 2 \).

(20) 5. Find \( \frac{dy}{dx} \).

a) \( y = 4x(5x^2 - 3)^7 \) **Answer** \( y' = 4(5x^2 - 3)^7 + 4x \cdot 7(5x^2 - 3^6) (10x) \).

b) \( y = \frac{\sin(4x)}{x^4 + 1} \) **Answer** \( y' = \cos(4x) \cdot \frac{16x^3 + 4x^2 - 12x^2 \sin(4x)}{(x^4 + 1)^2} \).

c) \( y = e^{x^2} + \cos(3x) \) **Answer** \( y' = \frac{2}{3} (e^{x^2} + \cos(3x)) (x^2 + 3 - x^2) \).

d) \( x^3 + x^2 y + 4y^3 = 6 \) **Answer** \( \frac{dy}{dx} \) the equation and get \( 3x^2 + 2xy + x^2 y' + 12y^2 y' = 0 \). Then solve for \( y' \) and get \( y' = -\frac{3x^2 - 2xy}{x^2 + 12y^2} \).
6. Here is a graph of \( y = f(x) \).

![Graph of \( y = f(x) \)]

a) Use this graph to sketch a graph of \( y = f'(x) \) on the axes below.

![Graph of \( y = f'(x) \)]

b) Are there \( x \)'s for which \( f(x) \) is not continuous? If there are, list them. **Answer** Yes, \( x = B \).

c) Are there \( x \)'s for which \( f(x) \) is not differentiable? If there are, list them. **Answer** Yes, \( x = B \) and \( x = A \).

d) Does \( y = f(x) \) seem to have any horizontal asymptotes? If it does, write equations for any lines which seem to be horizontal asymptotes. **Answer** Yes, \( y = E \).

e) Does \( y = f(x) \) seem to have any vertical asymptotes? If it does, write equations for any lines which seem to be vertical asymptotes. **Answer** Yes, \( x = B \).

7. Find all lines tangent to \( y = \frac{1}{x} \) which pass through the point \((-4,2)\).

**Answer** Since \( y' = -\frac{1}{x^2} \), and the slope of a line connecting \((x, y) = (x, \frac{1}{x})\) with \((-4,2)\) is \( \frac{\frac{1}{x} - 2}{x - (-4)} = \frac{\frac{1}{x} - 2}{x + 4} \), we know that \(-\frac{1}{x}\) should equal \( \frac{\frac{1}{x} - 2}{x + 4} \). If we cross-multiply, we get the equation \(-(x - (-4)) = x^2 (\frac{1}{x} - 2)\), and this becomes \(-x - 4 = x - 2x^2\) so that we need to solve \(2x^2 - 2x - 4 = 0\) or \(x^2 - x - 2 = 0\). Amazingly (or not, since it is a problem on an exam!) the left-hand side factors into \((x + 1)(x - 2)\) so the roots of the equation are \(-1\) and \(2\). When \(x = -1\), the point on \(y = \frac{1}{x}\) is \((-1, -1)\) and the slope is \(-1\), so that the tangent line is \((y + 1) = (-1)(x + 1)\).

When \(x = 2\), the point on \(y = \frac{1}{x}\) is \((2, \frac{1}{2})\) and the slope is \(-\frac{1}{2}\), so that the tangent line is \((y - \frac{1}{2}) = (-\frac{1}{2})(x - 2)\). To the right is a picture of the two lines and the curve, a hyperbola, drawn by Maple.